May 1, 2016

18.721 Assignment 9

This assignment is due Wednesday, May 11.

Let Y be a projective curve of genus $g > 0$. The module Ω_Y of differentials is isomorphic to $\mathcal{O}(K)$ for some divisor K of degree $2g-2$, called a canonical divisor, which is determined up to linear equivalence. Because the canonical module $\mathcal{O}(K)$ has nonzero global sections, there is an effective canonical divisor K.

If D is a divisor, the Serre dual of the module $\mathcal{O}(D)$ is $\mathcal{O}(K-D)$.

1. Suppose that $g = 2$.

(a) Determine all possible dimensions of $H^q(Y, \mathcal{O}(D))$, when D is an effective divisor of degree n.

(b) Let K be an effective canonical divisor. Then 1 is a global section of $\mathcal{O}(K)$, and there is also a nonconstant global section x. Prove that the pair of functions $(1, x)$ defines a morphism $Y \to \mathbb{P}^1$ that represents Y as a double cover of the projective line.

(c) Determine the number of branch points of this double covering.

2. Suppose that $g = 3$, an let K be and effective canonical divisor.

(a) Let $(1, x, y)$ be a basis for $H^0(Y, \mathcal{O}(K))$. Use Riemann-Roch for multiples of K to show that x, y satisfy a polynomial relation of degree at most 4.

(b) Let f be the morphism from Y to $\mathbb{P}^2$ defined by the rational functions $(1, x, y)$. Show that the image C of f is a plane curve of degree at most 4, and that if its degree is 4, then C is a smooth curve.