April 20, 2016

18.721 Assignment 8

This assignment is due Friday, April 29.

1. An ideal I of a domain R is locally principal if there are nonzero elements $s_1, ..., s_k$ that generate the unit ideal of R such that the localized ideal I_{s_i} of R_{s_i} is a principal ideal, for all i.

(a) Let R denote the polynomial ring $\mathbb{C}[x, y]$. Prove that every locally principal ideal I is principal.

Hint: If I is the principal ideal dR, then d will be the greatest common divisor of the elements of I. Look first at the case that the gcd of the elements of I is 1. But remember that R not a principal ideal domain. The gcd of two elements f, g is usually not a combination $af + bg$. For example, the gcd of x and y is 1.

(b) A module M over a domain R is *torsion-free* if for $a \in R$ and $m \in M$, $am = 0$ only when either a or m is zero. A torsion-free module M embeds as a submodule into the module of fractions $M_K = M \otimes_R K$, where K is the fraction field of R. Its *rank* is defined to be the dimension of M_K as K-vector space. Prove that a finite, torsion-free module M of rank 1 is isomorphic, as module, to an ideal of R.

(c) An R-module L is locally free, rank 1, if there are elements $s_1, ..., s_k$ that generate the unit ideal in R such that L_{s_i} is a free R_{s_i}-module of rank 1 for all i.. Locally free rank 1 modules are usually called invertible modules. Prove that any invertible R-module L is isomorphic, as module, to an ideal. Use this to extend part (a) to invertible R-modules.

2. Let $X = \mathbb{P}^2$. An $\mathcal{O}_X$-module $\mathcal{L}$ is invertible if its module of sections $\mathcal{L}(U^i)$ on the standard affine open set U^i is an invertible module for $i = 0, 1, 2$. Prove that every invertible $\mathcal{O}$-module $\mathcal{L}$ is isomorphic to a twisting module $\mathcal{O}(n)$.