March 10. 2016

18.721 Assignment 5

This assignment is due Friday, March 18

1. *(a map from a cusp curve to $\mathbb{P}^1$)* Let $A = \mathbb{C}[x,y]/(y^2 - x^3)$ and let X be the cusp curve $\operatorname{Spec} A$. Sending (x,y) to $(x-1, y-1)$ defines a morphism from the complement of the point $(1,1)$ in X to $\mathbb{P}^1$. Prove that this morphism can be extended to all of X.

2. *(blowing up a point in $\mathbb{P}^2$)* Consider the Veronese embedding of $\mathbb{P}^2_{xyz} \to \mathbb{P}^5_u$ by monomials of degree 2 defined by $(u_0, u_1, u_2, u_3, u_4, u_5) = (z^2, y^2, x^2, yz, xz, xy)$. If we drop the coordinate u_0, we obtain a map $\mathbb{P}^2 \xrightarrow{\varphi} \mathbb{P}^4$: $\varphi(x,y,z) = (y^2, x^2, yz, xz, xy)$ that is defined at all points except the point $q = (0,0,1)$. Prove that the inverse map φ^{-1} is everywhere defined, and that the fibre over q is a projective line.

3. *(lines in $\mathbb{P}^5$)* Let V be a vector space of dimension 5, and let $\mathbb{P}(W)$ denote the projective space of lines in $W = \bigwedge^2 V$, and let $\mathbb{G}$ denote the Grassmanian $G(2,5)$ of lines in $\mathbb{P}^4$. Prove

(a) There is a bijective correspondence between two-dimensional subspaces U of V and points of $\mathbb{P}(W)$ determined by nonzero decomposable vectors w. If (u_1, u_2) is a basis for U, the corresponding point of $\mathbb{P}(W)$ is $w = u_1 u_2$.

(b) A vector w in $\bigwedge^2 V$ is decomposable if and only if $ww = 0$.

(c) Exhibit defining equations for $\mathbb{G}$ in the space $\mathbb{P}(W)$.