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18.721 Comments on Assignment 4

1. Let X be the plane curve $y^2 = x(x-1)^2$, and let $A = \mathbb{C}[x,y]/(y^2 - x(x-1)^2)$ be its coordinate algebra. Let's use x, y also to denote the residues of those elements in A.

(a) Points of the curve can be parametrized by a variable t. Use the lines $y = t(x-1)$ to determine such a parametrization.

(b) Let $B = \mathbb{C}[t]$ and let T be the affine line $\operatorname{Spec}\mathbb{C}[t]$. *The parametrization gives us an injective homomorphism $A \to B$. Describe the corresponding morphism $T \to X$.*

(c) Let $s = x - 1$. Show that X is covered by the two localizations $X_s = \operatorname{Spec} A_s$ and $X_x = \operatorname{Spec} A_x$, where $A_s = A[s^{-1}]$ and $A_x = A[x^{-1}]$.

(a) Substituting $y = t(x-1)$ into the equation of the curve and cancelling gives $x = t^2$, and then $y = t(t^2 - 1)$.

(b) The morphism sends the point t of T to the point $(t^2, t(t^2-1)$ of X. It is an invetible map except at the points $t = \pm 1$, both of which are sent to $(1, 0)$, which is a node of X.

(c) To show this one meeds to show that the subsets $s = 0$ and $x = 0$ of X are disjoint. This is obvious.

2. (a locally principal ideal) Notation is as in the previous problem. The maximal ideal M of X at the point $p = (0, 0)$ is generated by the two elements x, y.

(a) Show that the localized ideal M_s of A_s, the ideal of A_s that is generated by M, is a principal ideal. Do the same for the localized ideal M_x.

(b) Using the ideal of $\mathbb{C}[t]$ that is generated by M, show that M is not a principal ideal.

(a) The ideal M is generated by x, y. Going over to A_s, we can write $x = y^2/s^2$. So y generates M. And in the ring A_x, M_x becomes the unit ideal because it contains the invertible element x.

(b) The ideal of $\mathbb{C}[t]$ generated by M is generated by the images of the elments x, y, which are $t^2, t(t^2 - 1)$. These two elements generate the principal ideal (t) of $\mathbb{C}[t]$. The only generators of the ideal (t) are scalar multiples of t. If M were a principal ideal, say gA, then $g = g(x, y)$ would generate (t) in $\mathbb{C}[t]$, which would imply that $g(t^2, t(t^2-1)) = ct$. Evaluating at $t = 1$ we would have $g(1, 0) = c$, and evaluating at $t = -1$ would give us $g(1, 0) = -c$. So there is no such polynomial g.

3. The cyclic group $< \sigma >$ of order n operates on the polynomial ring $R = \mathbb{C}[x,y]$, by $\sigma(x) = \zeta x$ and $\sigma(y) = \zeta y$, $\zeta = e^{2\pi i/n}$. Let A be the ring of invariants.

(a) Describe the invariant polynomials.

(b) Show that the polynomials $u_i = x^i y^{n-i}$, $i = 0, ..., n$, generate the ring A.

(c) Find generators for the ideal of relations among the generators u_i (the kernel of the homomorphism from the polynomial ring $\mathbb{C}[y_0, ..., y_n]$ to A that sends y_i to u_i).

(a) The space of invariant polynomials has as basis the monomials $m = v^i y^j$ such that $i + j \equiv 0$, modulo n.

(b) Let $m = v^i y^j$ be a monomial with $i + j \equiv 0$, modulo n. If $i \geq n$, u_n divides m. If $j >\geq n$, u_0 divides m. If i and j are less than n, then $m = u_i$.

(c) The claim is that the relations $u_i u_j = u_{i+1} u_{j-1}$, with $0 \leq i < n$ generate the ideal of all relations. We work modulo those relations.

Suppose given a monomial M of degree > 1 in u, and let i be the largest index of u_i that occurs in M. If $i < n$, and if there is another u_j in M with $j > 0$, we use the relations to replace $u_i u_j$ with $u_{i+1} u_{j-1}$ in M. When we can no longer do this, the u's that occur in M can be u_n, u_0, and at most one u_i with $0 < i < n$. Then $M = u_n^a u_i^b u_0^c = x^r y^s$, where $r = an + i$ and $s = cn + (n - i)$. The exponents r and s determine a, b, c, so the monomials M of this form are determined by the monomials in x, y.

4.*Let $A = \mathbb{C}[x_1, ..., x_n]$, and let $B = A[\alpha]$, where α is an element of the fraction field $\mathbb{C}(x)$ of A. Describe the fibres of the morphism $Y = \operatorname{Spec} B \to \operatorname{Spec} A = X$.*

Write $\alpha = g/h$ as a fraction of relatively prime polynomials. To determine the fibre, one solves the equation $h(p)y = g(p)$. If $h(p) \neq 0$, there is a unique soltution, so the fibre consists of a single point. If $h(p) = 0$ but $g(p) \neq 0$, there is no solution. The fibre is empty. If $h(p) = g(p) = 0$, then the equation is satisfied for all y. The fibre is an affine line.