February 26, 2016

18.721 Assignment 4

This assignment is due Friday, March 4

1. Let X be the plane curve $y^2 = x(x-1)^2$, and let $A = \mathbb{C}[x,y]/(y^2 - x(x-1)^2)$ be its coordinate algebra. Let's use x, y also to denote the residues of those elements in A.

(a) Points of the curve can be parametrized by a variable t. Use the lines $y = t(x-1)$ to determine such a parametrization.

(b) Let $B = \mathbb{C}[t]$ and let T be the affine line $\operatorname{Spec}\mathbb{C}[t]$. The parametrization gives us an injective homomorphism $A \to B$. Describe the corresponding morphism $T \to X$.

(c) Let $s = x - 1$. Show that X is covered by the two localizations $X_s = \operatorname{Spec} A_s$ and $X_x = \operatorname{Spec} A_x$, where $A_s = A[s^{-1}]$ and $A_x = A[x^{-1}]$.

2. (*a locally principal ideal)* Notation is as in the previous problem. The maximal ideal M of X at the point $p = (0,0)$ is generated by the two elements x, y.

(a) Show that the localized ideal M_s of A_s, the ideal of A_s that is generated by M, is a principal ideal. Do the same for the localized ideal M_x.

(b) Using the ideal of $\mathbb{C}[t]$ that is generated by M, show that M is not a principal ideal.

3. The cyclic group $< \sigma >$ of order n operates on the polynomial ring $R = \mathbb{C}[x,y]$, by $\sigma(x) = \zeta x$ and $\sigma(y) = \zeta y$, $\zeta = e^{2\pi i/n}$. Let A be the ring of invariants.

(a) Describe the invariant polynomials.

(b) Show that the polynomials $u_i = x^i y^{n-i}$, $i = 0, ..., n$, generate the ring A.

(c) Find generators for the ideal of relations among the generators u_i (the kernel of the homomorphism from the polynomial ring $\mathbb{C}[y_0, ..., y_n]$ to A that sends y_i to u_i).

4. Let $A = \mathbb{C}[x_1, ..., x_2]$, and let $B = A[\alpha]$, where α is an element of the fraction field $\mathbb{C}(x)$ of A. Describe the fibres of the morphism $Y = \operatorname{Spec} B \to \operatorname{Spec} A = X$.