

18.721 Comments on Assignment 3

1. To prove that the image in the dual plane of the set of smooth points of a curve of degree d is contained in a curve C^* , we used transcendence degree to conclude that there is a polynomial $G(t, s_0, s_1, s_2)$ such that $G(f, f_0, f_1, f_2)$ is identically zero, f being the defining equation for C . Use the following method to give an alternate proof: Determine the dimensions $N_r(k)$ of the spaces of polynomials of degree $\leq k$ in r variables, for $r = 3$ and $r = 4$. Show that $N_4(k) > N_3(kd)$ if k is large enough. Use counting of constants to show that there has to be a polynomial G that maps to zero by the substitution.

I don't know why I thought this wasn't suitable. Just senility. It is pretty simple.

2. The polynomial $f = 3(x^2 + y^2)z - x^3$ defines a cubic curve C with a node at $(0, 0, 1)$. Let C^* be its dual curve. Determine the degree of C^* and the numbers of flexes, bitangents, nodes, and cusps of C and C^* .

The cubic C has three ordinary flex points that can be computed explicitly, the point $(0, 1, 0)$ being one of them. Since C is a cubic, it has no bitangents. Therefore C^* has 3 cusps and no other singularities. Since C has no cusp, C^* has no flex points. It has a bitangent line that corresponds to the node of C .

The degree of C^* is 4. By definition, it is the number of intersections of C^* with a line in \mathbb{P}^* (counted with multiplicity). Let's project from the point $q = (1, 0, 0)$. The discriminant $D = \text{Discr}_x(f)$ will vanish at the images of the node $n = (0, 0, 1)$ and the points p of C whose tangent lines contain q . If the tangent line ℓ at p contains q , the line q^* intersects C^* at ℓ^* . So the degree of C^* is obtained from the degree D which is $3(3-1) = 6$ by subtracting the order of zero at the image of the node. The only complication is that the tangent to the flex at $(0, 1, 0)$ gives a double zero of D . That is doesn't affect the computation of the degree.

The discriminant is

$$D = -3^4 y^2 z^2 (4z^2 - 3y^2)$$

It has double zero at the image $y = 0$ of the node and another double zero at the image $z = 0$ of the flex.

(b) (extra credit) Determine the defining equation of C^* .

I think that the quartic is

$$9w^2v^2 + 36wuv^2 + 4wu^3 + 12(u^2 + v^2)^2 = 0$$

The equation is different from one found in Pset 2, because the equation of the cubic isn't symmetric with respect to permutation of the variables.

3. You are to describe singularities of curves that can be resolved by two blow ups.

The substitutions defining the blowups are $y = xv$ and $x = vu$.

With the given restrictions on C_2 , there are seven configurations that the curve could have. They are illustrated by the equations $v = u - 1$, $u = v - 1$, $v = (u - 1)^2$, $u = (v - 1)^2$, $v = u$, $v = u^2$, and $u = v^2$.

To find the images of these curves in the x, y -plane, we work backwards from the equations. using the fact that the maps $C_2 \rightarrow C_1 \rightarrow C_0$ are bijective except on the axes. For example, if C_2 is the curve $u = v^2$, the first step will be to write it in terms of x, v . To do this, we multiply by v , which is an invertible operation away from the axes. The result is $vu = v^3$, or $x = v^3$. Next, we multiply by x^3 , obtaining $x^4 = (xv)^3$, or $x^4 = y^3$. That is the equation of C_0 : a triple point. The case $v = u^2$, reduces as follows: $v^3 = v^2u^2$, $v^3 = x^2$, $x^3v^3 = x^5$, $y^3 = x^5$, a different triple point.

I didn't expect you to classify these singularities