February 19, 2016

18.721 Assignment 3

This assignment is due Friday, February 26

1. To prove that the image in the dual plane of the set of smooth points of a curve of degree d is contained in a curve C^*, we used transcendence degree to conclude that there is a polynomial $G(t, s_0, s_1, s_2)$ such that $G(f, f_0, f_1, f_2)$ is identically zero, f being the defining equation for C. Use the following method to give an alternate proof: Determine the dimensions $N_r(k)$ of the spaces of polynomials of degree $\leq k$ in r variables, for $r = 3$ and $r = 4$. Show that $N_4(k) > N_3(kd)$ if k is large enough. Use counting of constants to show that there has to be a polynomial G that maps to zero by the substitution.

Note: This method doesn't give a good bound for the degree of C^*. One reason is that the partial derivatives have degree $d - 1$. Another may be that f and its derivatives are related by Euler's Formula. It is tempting try using Euler's Formula to help compute the equation of C^*, but I haven't succeeded in getting anywhere that way. If you have an idea, please let me know.

2. The polynomial $f = 3(x^2 + y^2)z - x^3$ defines a cubic curve C with a node at $(0, 0, 1)$. Let C^* be its dual curve. Determine the degree of C^* and the numbers of flexes, bitangents, nodes, and cusps of C and C^*.

(b) (extra credit) Determine the defining equation of C^*.

3. You are to describe singularities of curves that can be resolved by two blow ups.

Start with a curve C_0 in the the affine x, y-plane P_0 and that contains the origin p. Let the first blowup be the x, v-plane P_1, with $y = xv$, and let the second blowup be the u, v-plane P_2, with $x = uv$. So P_2 contains two lines that map to the origin in P_0, the u-axis U and the v-axis V. The inverse image of C_0 in P_2 will be the union of U, V, and a curve C_2 called the *proper inverse image*.

Suppose that C_2 is smooth above the origin p of P_0, that it meets the $U \cup Z$ in just one point, and that it has at most an ordinary tangency with either of the axes U and V. There are several ways that C_2 might be situated with respect to the axes. Describe them with pictures. Then find sample equations for C_0, and determine the multiplicity of C_0 at p in each case.