18.721 Comments on Assignment 2

1. (a) Let f a homogeneous polynomial in x, y, z, not divisible by z. Prove that f is irreducible if and only if $f(x, y, 1)$ is irreducible.
(b) Prove that most nonhomogeneous polynomials in two or more variables are irreducible.

For (b), I wanted you to show that the product gh of polynomials of degrees i, j with $i+j \le d$ depend on fewer parameters than the polynomials f of degree $\le d$.

2. Let $f(x, y, z)$ and $g(x, y, z)$ be homogeneous polynomials of degrees m and n, with no common factor, let R be the polynomial ring $\mathbb{C}[x, y, z]$, and let $A = R/(f, g)$.

(a) Show that the sequence

$$0 \to R_{(1)} \xrightarrow{(-g,f)} R^2_{(2)} \xrightarrow{(f,g)^t} R_{(3)} \to A_{(4)} \to 0$$

is exact.

(I've put subscripts (i) into the sequence to locate the terms.)

We are given that A is the quotient of R, modulo the ideal generated by f, g. Therefore the sequence is exact at (3) and (4).

Exactness at (1): we must show that the map $(-g, f)$ is injective, and this is true because R is a domain.

Exactness at (2): We have to show that the kernel of $(f, g)^t$ is equal to the image of $(-g, f)$. Let (u, v) be an element of R^2 in the kernel of $(f, g)^t$, i.e., such that $uf + vg = 0$. Then $uf = -vg$. Since f and g have no common factor, f divides u and g divides v, and moreover, if $u = gp$, then $v = fp$. Therefore $(u, v) = (-g, f)p$. So (u, v) is in the image of $(-g, f)$.

(b) (algebraic version of Bézout's Theorem) Because f and g are homogeneous, A inherits a grading by degree, i.e., $A = A_0 \oplus A_1 \oplus \cdots$, where A_n is the image in A of the space R_n consisting of the homogeneous polynomials of degree n together with 0. Prove that $\dim A_k = mn$ for all sufficiently large k.

Multiplication by f raises degree by m and multiplication by g raises degree by n. Working degree by degree in the exact sequence, the following will be an exact sequence of vector spaces:

$$0 \to R_{k-m-n} \xrightarrow{(-g,f)} R_{k-m} \oplus R_{k-n} \xrightarrow{(f,g)^t} R_k \to A_k \to 0$$

Therefore

$$\dim A_k - \dim R_k + (\dim(R_{k-m} + \dim R_{k-n}) - \dim R_{k-m-n} = 0.$$

One determines that $\dim R_k = \binom{k+2}{2}$. Then

$$\dim A_k = \binom{k+2}{2} - \binom{k-m+2}{2} - \binom{k-n}{2} + \binom{k-m-n+2}{2}$$

This works out to mn.

4. Exhibit an irreducible homogeneous polynomial $f(x,y,z)$ of degree 4 whose locus of zeros is a curve with three cusps.

We choose coordinates so that the cusps are the coordinate vertices. Since the cusps are double points, the coefficients of $x^4, x^3y, x^3z,\ y^4, xy^3, y^3z,\ z^4, xz^3, yz^3$ in f will be zero, leaving us with a combination of $x^2y^2, x^2z^2, y^2z^2,\ x^2yz, xy^2z, xyz^2$.

Let's try normalizing the first three coefficients to 1: $f = (x^2y^2 + x^2z^2 + y^2z^2) + (ax^2yz + bxy^2z + cxyz^2)$. Setting $z = 1$, the quadratic term $x^2 + y^2 + cxy$ is supposed to be a square. So $c = \pm 2$. Similarly, $a = \pm 2$ and $b = \pm 2$. The polynomial $(x^2y^2 + x^2z^2 + y^2z^2) \pm 2(x^2yz + xy^2z - xyz^2)$ is one possibility. However, when the sign is $+$, this polynomial is equal to $(xy + xz + yx)^2$. Not irreducible, therefore not good. We try the sign $-$:

$$f = (x^2y^2 + x^2z^2 + y^2z^2) - 2(x^2yz + xy^2z - xyz^2)$$

The symmetry of this polynomial helps show that it is irreducible. If $ax + by + cz$ divides f, then so do $ax + cy + bz$ and $bx + cy + az$. Thinking this through, one sees that the only possible linear factor is $x + y + z$. But the point $(1, -1, 0)$ lies on that line and not on $f = 0$. So there is no linear factor. Suppose that $f = gh$ is a product of quadratic polynomials. If the coefficient of x^2 in g were nonzero, then because there is no opccurence of x^3 or x^4 in f, h would be a polynomial in y, z, and it would factor into linear factors. So x^2 has zero coefficient in g, as do y^2 and z^2. Therefore g and h must be combinations of xy, xz, yz, and by symmetry they must be $xy + xz + yz$, up to scalar factor. We've already seen what the square of this polynomial is.