Spring 2016

18.721 Assignment 2

This assignment is due friday, February 19

1. (a) Let f a homogeneous polynomial in x, y, z, not divisible by z. Prove that f is irreducible if and only if $f(x, y, 1)$ is irreducible.
(b) Prove that most nonhomogeneous polynomials in two or more variables are irreducible.

2. Let $f(x, y, z)$ and $g(x, y, z)$ be homogeneous polynomials of degrees m and n, with no common factor, let R be the polynomial ring $\mathbb{C}[x, y, z]$, and let $A = R/(f, g)$.
(a) Show that the sequence

$$0 \to R \xrightarrow{(-g,f)} R^2 \xrightarrow{(f,g)^t} R \to A \to 0$$

is exact.
(b) *(algebraic version of Bézout's Theorem)* Because f and g are homogeneous, A inherits a grading by degree, i.e., $A = A_0 \oplus A_1 \oplus \cdots$, where A_n is the image in A of the space R_n consisting of the homogeneous polynomials of degree n together with 0. Prove that $\dim A_k = mn$ for all sufficiently large k.

3. Let p be a cusp of the curve C defined by a homogeneous polynomial f. Prove that there is just one line ℓ through p such that the restriction of f to ℓ has as zero of order > 2, and that the order of zero for this line is precisely 3.

4. Exhibit an irreducible homogeneous polynomial $f(x, y, z)$ of degree 4 whose locus of zeros is a curve with three cusps.