18.721 Comments on Assignment 1

2. Let f and g be irreducible homogeneous polynomials in x, y, z. Prove that if the loci $\{f = 0\}$ and $\{g = 0\}$ are equal, then $g = cf$.

We write f and g as polynomials in z whose coefficients are polynomials in x, y, and we embed $R = \mathbb{C}[x, y, z]$ into the ring $F[z]$, where $F = \mathbb{C}(x, y)$.

Suppose that g isn't a constant multiple of f. Then because these polynomials are irreducible, they have no common factor in R. They can't have a common factor in $F[z]$ either. If h was a common factor in $F[z]$, one could clear denominators to make h an element of R, and replace h by an irreducible factor in R that involves z. Then since h divides f in $F[z]$, it divides in R too.

This being so, one can write $pf + qg = 1$, with p, q in $F[z]$. Clearing denominators give an equation in R of the form $\tilde{p}f + \tilde{q}g = d(x, y)$. Then for any point (x_0, y_0) such that $d(x_0, y_0) \neq 0$, $f(x_0, y_0, z)$ and $g(x_0, y_0, z)$ have no common zeros.

4. Prove that a plane cubic curve can have at most one singular point.

Suppose that the cubic curve C is singular. We choose coordinates so that the singular point is $p = (0, 0, 1)$. Let the equation for C be $f(x, y, z) = 0$. Then $f(p) = 0$ because $p \in C$, and $f_x(p) = f_y(p) = 0$ because p is a singular point This impies that the coefficients of the monomials z^3, xz^2, yz^2 in f are zero. If there were another singular point, we could put it at $q = (1, 0, 0)$, and the same reasoning would show that the coefficients of x^3, x^2y, x^2z in f are zero. Then f would be a combination of y^3, xy^2, y^2z, xyz, and would be divisible by y, contradicting irreducibility.

6. Let C be a smooth cubic curve in $\mathbb{P}^2$, and let p be a flex point of C. Choose coordinates so that p is the point $(0, 1, 0)$ and the tangent line to C at p is the line $\{z = 0\}$.
(a) *Show that the coefficients of x^2y, xy^2, and y^3 in the defining polynomial f of C are zero.*
(b) *Show that with a suitable choice of coordinates, one can reduce the defining polynomial to the form $f = y^2z + x^3 + axz^2 + bz^3$, where $x^3 + ax + b$ is a polynomial with distinct roots.*
(c) *Show that one of the coefficients a or b can be eliminated.*

(b) With coordinates as indicated, the cubic polynomial has the form $f(x, y, z) = *y^2z + \ell(x, z)yz + c(x, z)$, where ℓ and c are homogeneous linear and cubic, respectively. The coefficient of y^2z will be nonzero, and can be normalized to 1. Then $f = 0$ is a quadratic equation in y. Completing the square by the substitution $y \to y - \frac{1}{2}\ell$ eliminates the linear term, leaving us with $y^2z + c'(x, z)$. The quadratic term in z can be eliminated by a substitution $x \to x + *z$.

(c) Since C is smooth, a and b aren't both zero. Unless $b = 0$, scaling can be used to replace b by 1. If $b = 0$, one can make $a = 1$ instead.