VANISHING OF THE DISCRIMINANT

This is an outline of the Wednesday Feb 10 and Friday, February 12 classes.

about the resultant

Lemma 1. *Let $f(x), g(x), h(x)$ be monic polynomials in one variable x.*
(i) $\text{Res}(f, gh) = \text{Res}(f, g)\,\text{Res}(f, h)$.
(ii) *If* $\deg f \geq \deg g$, *then* $\text{Res}(f, g) = \text{Res}(f + g, g)$.

proof. **(i)** follows from the formula $\text{Res}(f, g) = \prod(\alpha_i - \beta_j)$, where α_i and β_j are the roots of f and g, respectively.

(ii) The coefficients of $f + g$ are obtained by adding the coefficients of g to those of f. This being so, the resultant matrix for $\text{Res}(f + g, g)$ is obtained from the resultant matrix for $\text{Res}(f, g)$ by elementary row operations of adding multiples of one row to another. The determinant is unchanged.

The resultant matrix for $\text{Res}(f+g, g)$ is shown below for the case that $f = a_0x^2+a_1x+a_2$ and $g = b_0x+b_1$.

$$\begin{pmatrix} a_0 & a_1 + b_0 & a_2 + b_1 \\ b_0 & b_1 & \cdot \\ \cdot & b_0 & b_1 \end{pmatrix}$$

using the Implicit Function Theorem

Let $f(t, x) = x^m + a_1x^{m-1} + \cdots + a_m$ be a monic polynomial in x, whose coefficients a_i are polynomials in t, and let $\overline{f}(x) = f(0, x)$. Suppose that $x = 0$ is a simple root of $\overline{f}$. Then the derivative $\frac{d\overline{f}}{dx}$ isn't zero at $x = 0$, and the partial derivative $\frac{\partial f}{\partial x}$ isn't zero at $t, x = 0, 0$. The Implicit function Theorem tells us that we can solve the equation $f(t, x) = 0$ uniquely for x as an analytic function $\phi(t)$, defined for small t, and with $\phi(0) = 0$. Then $f(t, \phi(t)) = 0$ for small t.

(The Implicit Function Theorem is in Rudin. I think he proves only that ϕ continuous, not that it is analytic, i.e., that it can be expressed as a power series in t. This isn't very important. continuity would be enough for our purposes.)

Lemma 2. *Let $f(t, x)$ be as above, and let P be the ring of functions in t that are defined and analytic for small t. In the ring $P[x]$ of polynomials in x whose coefficients are analytic functions of t, the polynomial $x - \phi(t)$ divides $f(t, x)$.*

proof. One can do division with remainder by a monic polynomial when coefficients are in any ring. In $P[x]$, we divide $f(t, x)$ by $x - \phi(t)$, which is a monic polynomial in x:

$$f(t, x) = (x - \phi)q(t, x) + r(t)$$

The remainder $r(t)$ is an element of P because its degree in x is zero. Substituting $x = \phi$ shows that the remainder is zero. □

order of vanishing of the resultant

Let $f(t, x) = x^m + a_1x^{m-1} + \cdots + a_m$ and $g(t, x) = x^n + b_1x^{n-1} + \cdots + b_n$ be monic polynomials in x, whose coefficients a_i and b_j are polynomials in t, and let $\overline{f}(x) = f(0, x)$ and $\overline{g}(x) = g(0, x)$.

The resultant $\text{Res}_x(f, g)$, computed with respect to the variable x, is a polynomial in t. Let's denote it by $R(t)$. The resultant $\text{Res}_x(\overline{f}, \overline{g})$ is obtained by evaluating $R(t)$: $\text{Res}_x(\overline{f}, \overline{g}) = R(0)$.

Proposition 1. *Suppose that $x = 0$ is a simple root of $\overline{f}$ and $\overline{g}$, and that this is their only common root. Suppose also that the loci $C : \{f = 0\}$ and $D : \{g = 0\}$ intersect transversally at the origin. Then the resultant $R(t)$ has a simple zero at $t = 0$.*

proof. We solve $f = 0$ near the origin, say $x = \phi(t)$, and we divide: $f(t, x) = (x - \phi)q(t, x)$. Similarly, $g(t, z) = (x - \psi)s(t, x)$. Then in a neighborhood of the origin, C is the locus $x = \phi(t)$ and D is the locus $x = \psi(t)$. Since these loci intersect transversally, $\frac{d\phi}{dt}(0) \neq \frac{d\psi}{dt}(0)$.

Because $x = 0$ is a simple root of $\overline{f}$ and $\overline{g}$, $\overline{q} = q(0, x)$ has no root in common with $\overline{s} = s(0, x)$, and $x = 0$ isn't a root of $\overline{q}$ or $\overline{s}$. By Lemma 1, $\text{Res}(f, g)$ is the product of four terms:

$$\text{Res}((x - \phi), (x - \psi)) \cdot \text{Res}((x - \phi), s) \cdot \text{Res}(q, s) \cdot \text{Res}(q, (x - \psi))$$

Here $\text{Res}((x - \phi), (x - \psi))$ is the only term that vanishes at $t = 0$. Therefore the order of vanishing of $\text{Res}(f, g)$ is the same as that of

$$\text{Res}((x - \phi), (x - \psi)) = \det \begin{pmatrix} 1 & -\phi \\ 1 & -\psi \end{pmatrix} = \phi(t) - \psi(t)$$

and since $\frac{d\phi}{dt}(0) \neq \frac{d\psi}{dt}(0)$, the order of vanishing is 1. □

application to B'ezout's Theorem

Theorem 1. *Let C and D be plane projective curves of degrees m and n, respectively, and suppose C meets D transversally at every point of intersection. Then the number of intersections is precisely mn.*

proof. Let $f(x, y, z)$ and $g(x, y, z)$ be the irreducible homogeneous polynomials of degrees m and n, whose zero loci are C and D, respectively.

We project $\mathbb{P}^2$ to $\mathbb{P}^1$, using a generic center of projection q. We choose coordinates so that $q = (0\,0\,1)$. This point will not lie on C, and therefore $f(0, 0, 1) \neq 0$, which means that the coefficient of z^m in f isn't zero. We normalize that coefficient to 1, so that f becomes a monic polynomial in z, say $f = z^m + a_1 z^{m-1} + \cdots + a_m$, where a_i is a homogeneous polynomial of degree i in x, y. Similarly, we may assume that $g = x^n + b_1 x^{n-1} + \cdots + b_n$, where b_i is homogeneous of degree i in x, y.

The discriminant $\text{Res}_z(f, g)$ with respect to the variable z will be a homogeneous polynomial of degree mn in x, y. Let's denote it by $R(x, y)$. When we evaluate at a point (x_0, y_0) of $\mathbb{P}^1$, $R(x_0, y_0)$ will be the resultant $\text{Res}_z(\overline{f}, \overline{g})$, where $\overline{f}$ and $\overline{g}$ are the one-variable polynomials $f(x_0, y_0, z)$ and $g(x_0, y_0, z)$. So (x_0, y_0) will be a zero of $R(x, y)$ if and only if $f(x_0, y_0, z)$ and $g(x_0, y_0, z)$ have a common root z_0. Then the point (x_0, y_0, z_0) is a zero of f and of g, so it will be a point of intersection of C and D. Conversely, the projection (x_0, y_0) of an intersection point (x_0, y_0, z_0) will be a zero of $R(x, y)$.

The plan is to show that, provided that q is generic, the resultant $R(x, y)$ polynomial has simple zeros. Then the number of zeros will be mn, and this will also be the number of points of intersection $|C \cap D|$. What we require is:

(i) q isn't a point of C or D, and
(ii) the line L through q and an intersection point p doesn't contain any other intersection.
(iii) the line L through q and an intersection point p isn't tangent to C or D at any point.

Condition (i) was discussed above, (ii) tells us that the projections of distinct intersection points are distinct, and (iii) will be used below.

It suffices to show that the projection $\widetilde{p}$ of one of the intersection points, say p, is a simple zero of the resultant. This will follow from Proposition 1. We adjust coordinates so that $p = (1\,0\,0)$, keeping $q = (0\,0\,1)$. This allows us to set $x = 1$ and to work in the affine y, z-plane U, an open subset of $\mathbb{P}^2$. In affine coordinates,

p becomes the origin $(y,z) = (0\,0)$ and its projection $\widetilde{p}$ is the point $y = 0$ of the affine y-line. The defining equations for C and D in U are $F(y,z) = f(1,y,z)$ and $G(y,z) = g(1,y,z)$, and the line L through q and p becomes the z-axis $y = 0$.

Let $\overline{F} = F(0,z)$ and $\overline{G} = G(0,z)$. Because L isn't a tangent line to C or D, $z = 0$ is a simple root of $\overline{F}$ and of $\overline{G}$. Proposition 1 applies to show that the resultant has a simple zero at $y = 0$. □

Euler Characteristic of a smooth plane curve

Let C: $f(x,y,z) = 0$ be a smooth plane curve of degree d. The Euler Characteristic is described in Chapter 1 of the notes. It is computed as $e(C) = v - e + f$, where v, e, f are the numbers of vertices, edges, and faces of a topological triangulation of C. To compute this Euler characteristic, we project C generically to $\mathbb{P}^1$. With notation as before, the discriminant $\mathrm{Discr}_z(f) = D(x,y)$, computed with respect to the variable z, will vanish at the points $\widetilde{p} = (x_0, y_0)$ such that the line $L = \{(x_0, y_0, z)\}$ meets C in fewer than d points, which means that the one-variable polynomial $f(x_0, y_0, z)$ has a multiple root. The zeros of D are the *branch points* of the covering $C \to \mathbb{P}^1$.

The discriminant $D(x,y)$ is a homogeneous polynomial in x, y of degree $d(d-1)$. The plan is to show that, provided that the projection is generic,

(a) D has $d(d-1)$ simple zeros, and
(b) at a zero $\widetilde{p} = (x_0, y_0)$ of D, the polynomial $f(x_0, y_0, z)$ has one double root and $d - 2$ simple roots.

When this is shown we will be able to compute the Euler Characteristic of C as follows: We trangulate $\mathbb{P}^1$ in such a way that the branch points, the zeros of D, are among the vertices, and we triangulate C using the inverse image of this triangulation. Since most points of $\mathbb{P}^1$ are covered by d points of C, there will be d faces of C lying over each face of $\mathbb{P}^1$, d edges over each edge of $\mathbb{P}^1$, and d vertices over each vertex of $\mathbb{P}^1$ except for the vertices that are branch points. Over the branch points, there will be only $d-1$ vertices. The Euler Characteristic of $\mathbb{P}^1$ is 2. Therefore Euler Characteristic of C is

$$e(C) \;=\; d\,e(\mathbb{P}^1) - d(d-1) \;=\; 2d - d(d-1) \;=\; 3d - d^2$$

To prove (a) and (b), we choose the center of projection $q = (0\,0\,1)$ so that, if a line L through q is tangent to C at a point $p = (x_0, y_0, z_0)$, then

(i) p is not a flex point of C, and
(ii) L is not tangent to C at a second point.

These conditions tell us that C and L have a simple contact at p, therefore that $\overline{f} = f(x_0, y_0, z)$ has a double root at $z = z_0$. Its other roots are simple roots. Then (b) will be true. We want to use Proposition 1 to prove (a).

Let g denote the partial derivative $\frac{\partial f}{\partial z}$. The discriminant D is the resultant of f and g. Unfortunately, Proposition 1 doesn't apply directly, Though $\overline{g} = g(x_0, y_0, z)$ has a simple root at $z = z_0$, $\overline{f}$ has a double root there. The trick is to replace f by $h = f + g$. Then $\overline{h} = h(x_0, y_0, z)$ has a simple root at z_0, and the zero loci $\{g = 0\}$ and $\{h = 0\}$ intersect transversally at $p = (x_0, y_0, z_0)$. Lemma 1 (ii) shows that the discriminant $\mathrm{Discr}_z(f)$ is equal to the resultant $\mathrm{Res}_z(h, g)$. Since $\overline{f}$ has only one multiple root, z_0 is the only common root of $\overline{f}$ and $\overline{g}$, and the same is true of $\overline{h}$ and $\overline{g}$. Now Proposition 1 applies. □