BACKGROUND

Things may be added here as the semester goes on.

The section on Tensor Products has been moved here.

0.1 Exact Sequences

exactseq

A sequence

$$\cdots \to V^{n-1} \xrightarrow{d^{n-1}} V^n \xrightarrow{d^n} V^{n+1} \xrightarrow{d^{n+1}} \cdots$$

of homomorphisms of abelian groups is *exact* if the image of d^{n-1} is equal to the kernel of d^n.

For example, a sequence

$$0 \to V \xrightarrow{d} V'$$

is exact if and only if the map d is injective. A sequence

$$V \xrightarrow{d} V' \to 0$$

is exact if and only if d is surjective, and a sequence

$$0 \to V \xrightarrow{d} V' \to 0$$

is exact if and only if d is bijective.

Any homomorphism $V \xrightarrow{d} V'$ can be embedded into an exact sequence

$$0 \to K \to V \xrightarrow{d} V' \to C \to 0,$$

where K and C are the kernel and cokernel of d, respectively.

A *short exact sequence* is an exact sequence

$$0 \to V \to V' \to V'' \to 0.$$

The statement that this sequence is exact asserts that the map $V \to V'$ is injective, and that V'' is isomorphic to the quotient group V'/V.

kerfunct **0.1.1. Lemma.** *(functorial property of the kernel and cokernel) Suppose given a (commutative) diagram of homomorphisms of abelian groups*

$$\begin{array}{ccc} V & \longrightarrow & V' \\ {\scriptstyle f}\big\downarrow & & {\scriptstyle f'}\big\downarrow \\ W & \longrightarrow & W' \end{array}$$

Let K and C be the kernel and cokernel of f, and let K' and C' be defined analogously. There are canonical homomorphisms $K \to K'$ and $C \to C'$. □

snake **0.1.2. Proposition.** *Suppose given a (commutative) diagram*

$$\begin{array}{ccccccccc} (0 & \longrightarrow &)V & \xrightarrow{f} & V' & \longrightarrow & V'' & \longrightarrow & 0 \\ & & {\scriptstyle f}\big\downarrow & & {\scriptstyle f'}\big\downarrow & & {\scriptstyle f''}\big\downarrow & & \\ 0 & \longrightarrow & W & \longrightarrow & W' & \underset{g}{\longrightarrow} & W''(& \longrightarrow & 0) \end{array}$$

whose rows are short exact sequences. The maps indicated in parentheses are optional. Let K, K', K'' and C, C', C'' denote the kernels and cokernels of f, f', f'', respectively.
(i) *(left exactness of the kernel) The kernels form an exact sequence*

$$K \to K' \to K''.$$

If f is injective, the sequence $0 \to K \to K' \to K''$ is exact.
(ii) *(right exactness of the cokernel) The cokernels form and exact sequence*

$$C \to C' \to C''.$$

If g is surjective, the sequence $C \to C' \to C'' \to 0$ is exact.
(iii) ***Snake Lemma.*** *There is a canonical homomorphism $K'' \xrightarrow{d} C$ that combines with the above sequences to form an exact sequence*

$$(0 \longrightarrow)\ K \to K' \to K'' \xrightarrow{d} C \to C' \to C''\ (\longrightarrow 0).$$

□

0.2 Complexes

complexes

A *complex* $R^\bullet$ of abelian groups is a sequence of homomorphisms

$$\cdots \to R^{n-1} \xrightarrow{d^{n-1}} R^n \xrightarrow{d^n} R^{n+1} \xrightarrow{d^{n+1}} \cdots$$

with the property that the composition $d^n d^{n-1}$ of adjacent maps is zero, i.e., that the image of d^{n-1} is contained in the kernel of d^n. An exact sequence is a complex.

A *map* $R^\bullet \xrightarrow{\varphi} R'^\bullet$ of complexes is a collection of homomorphisms $R^n \xrightarrow{\varphi^n} R'^n$ making a commutative diagram

$$\begin{array}{ccccccccc} \longrightarrow & R^{n-1} & \xrightarrow{d^{n-1}} & R^n & \xrightarrow{d^n} & R^{n+1} & \longrightarrow & \cdots \\ & {\scriptstyle \varphi^{n-1}}\big\downarrow & & {\scriptstyle \varphi^n}\big\downarrow & & {\scriptstyle \varphi^{n+1}}\big\downarrow & & \\ \longrightarrow & R'^{n-1} & \xrightarrow{d'^{n-1}} & R'^n & \xrightarrow{d'^n} & R'^{n+1} & \longrightarrow & \ldots \end{array}$$

A sequence of maps of complexes

$$\cdots \to R^\bullet \xrightarrow{\varphi} R'^\bullet \xrightarrow{\psi} R''^\bullet \to \cdots$$

is called *exact* if the sequence

$$\cdots \to R^q \xrightarrow{\varphi^q} R'^q \xrightarrow{\psi^q} R''^q \to \cdots$$

is exact for every q.

The *cohomology* $\mathbf{h}^q$ of a complex $R^\bullet$ is defined by

$$\mathbf{h}^q(R^\bullet) = (\ker\, d^q)/(\mathrm{im}\, d^{q-1}).$$

chiisaltsum

0.2.1. Proposition. *Let $0 \to V^0 \to V^1 \to \cdots \to V^n \to 0$ be a complex of finite dimensional vector spaces. Then*

$$\sum (-1)^q \dim V^q = \sum (-1)^q \dim \mathbf{h}^q(V^\bullet).$$

□

0.3 Categories

categ

A *category* consists of "objects" and "morphisms" between objects, with certain properties that are described below. Morphisms may also be called maps, and often the objects are sets (with some additional structure) and the morphisms are maps of sets, but they needn't be. Two examples of this intuitive concept are:

- the category (ab): Its objects are abelian groups and its morphisms are group homomorphisms.
- the category (top spaces): Its objects are topological spaces and its morphisms are continuous maps.

As one sees from these examples, the objects and morphisms may form sets large enough to that, if handled badly, might cause logical problems related to Russell's Paradox. This will not be an issue for us, so we ignore logical problems and say that a category $\mathcal{C}$ consists of a set (objects)$_\mathcal{C}$ of objects and a set (morphisms)$_\mathcal{C}$ of morphisms.

There are two objects associated to a morphism f: its *domain* and its *range*. If X and Y are the domain and range of a morphism f, one writes $X \xrightarrow{f} Y$. (This is symbolic notation, since the morphisms don't need to be maps).

The domains and ranges of the morphisms are described by maps of sets

$$(\text{morphisms})_\mathcal{C} \xrightarrow{domain} (\text{objects})_\mathcal{C}$$

and

$$(\text{morphisms})_\mathcal{C} \xrightarrow{range} (\text{objects})_\mathcal{C}$$

Furthermore, one can compose morphisms: To a pair f, g of morphisms such that $X \xrightarrow{f} Y \xrightarrow{g} Z$, i.e., such that the range of f is the domain of g, there is associated a *composed morphism* $X \xrightarrow{g \circ f} Z$. There are two axioms for this law of composition:

- (*associative law*)

  Given morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$, the composed morphisms $h \circ (g \circ f)$ and $(h \circ g) \circ f$ are equal.

- (*identity*)

  For every object Y, there is an *identity* morphism $Y \xrightarrow{id_Y} Y$ with domain and range Y, such that, if $X \xrightarrow{f} Y$, then $id_Y \circ f = f$ and if $Y \xrightarrow{g} Z$, then $g \circ id_Y = g$.

You needn't worry too much about the definition. Just remember that a category consists of some objects and some morphisms between objects. There is nothing peculiar about the rest.

0.4 Functors

functor

The concept of a functor is also intuitive. A *functor* F from a category $\mathcal{C}$ to another category $\mathcal{D}$,

$$\mathcal{C} \xrightarrow{F} \mathcal{D},$$

is a map that sends objects to objects and morphisms to morphisms. So it consists of a pair of maps

$$\text{(objects)}_{\mathcal{C}} \to spoint\text{(objects)}_{\mathcal{D}}$$

and

$$\text{(morphisms)}_{\mathcal{C}} \to \text{(morphisms)}_{\mathcal{D}}$$

These maps are required to be compatible with the category structures:

- If $X \xrightarrow{f} Y$ in $\mathcal{C}$, then $F(X) \xrightarrow{F(f)} F(Y)$,
- If $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{C}$, then $F(g \circ f) = F(g) \circ F(f)$,
- $F(id_Y) = id_{F(Y)}$.

For example, sending a topological space to its underlying set is a functor (top spaces) $\to$ (sets). Sending a pointed, path-connected topological space X to its fundamental group $\pi_1(X)$ is a functor from the category $(pt.conn.top)$ to the category $(groups)$. As is the case for these examples, it often becomes clear what the functor is supposed to do to morphisms, once the map on objects has been described.

dual **(0.4.1) contravariant functors**

There is a closely related concept, that of a *contravariant functor*, written as

$$\mathcal{C}^\circ \xrightarrow{F} \mathcal{D}.$$

A contravariant functor is like a functor except that it reverses the direction of a morphism: If $X \xrightarrow{f} Y$ in $\mathcal{C}$, then, applying F, one obtains a morphism $F(X) \xleftarrow{F(f)} F(Y)$ in the opposite direction. The fact that F reverses arrows is indicated by the superscript $\circ$. As before, F is required to be compatible with composition of morphisms, and identity morphisms are sent to identity morphisms. The only thing to note is that, because F reverses arrows, it also reverses the order of composition of morphisms: If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are morphisms in $\mathcal{C}$, then $F(X) \xleftarrow{F(f)} F(Y) \xleftarrow{F(g)} F(Z)$, so the requirement is that $F(g \circ f) = F(f) \circ F(g)$.

For example, let *(vector spaces)* denote the category whose objects are complex vector spaces and whose morphisms are linear transformations. Sending a vector space V to its dual space V^* is a contravariant functor *(vector spaces)*$^\circ \to$ *(vector spaces)* from the category to itself. Sending a topological space X to its qth integer cohomology group $H^q(X, \mathbb{Z})$ is a contravariant functor (top spaces)$^\circ \to$ (ab).

When $\mathcal{C}^\circ \xrightarrow{F} \mathcal{D}$ is a contravariant functor and f is a morphism in $\mathcal{C}$, the map $F(f)$ is often denoted (ambiguously) by f^*, the superscript $*$ indicating that the direction is reversed. (I don't know why one needs two notations, $\circ$ and $*$, for the reversal of arrows.)

For emphasis, an ordinary functor is sometimes called a *covariant functor*. A contravariant functor $\mathcal{C}^\circ \xrightarrow{F} \mathcal{D}$ can be turned into a covariant functor by changing the category $\mathcal{C}$ to its dual category. The *dual category* $\mathcal{C}^\circ$ of a category $\mathcal{C}$ is a category whose objects correspond bijectively to the objects of $\mathcal{C}$, and if the objects X°, Y° of $\mathcal{C}^\circ$ correspond to the objects X, Y of $\mathcal{C}$, the morphisms $X^\circ \xleftarrow{f^\circ} Y^\circ$ correspond bijectively to morphisms $X \xrightarrow{f} Y$ in $\mathcal{C}$ in the other direction. For instance, the dual category of the category of finite-type domains is the category of affine varieties.

0.5 The Hom Functors

homfnctr .

If X and Y are objects of a category $\mathcal{C}$, we often denote the set of morphisms $X \to Y$ in $\mathcal{C}$ by $\mathrm{Hom}(X, Y)$ or by (X, Y).

Let $Z \xrightarrow{\alpha} Z'$ be a morphism in $\mathcal{C}$ and let X be another element of $\mathcal{C}$. Composition on the left with α defines a map $(Z, X) \leftarrow (Z', X)$, that sends a morphism $Z' \xrightarrow{f} X$ to $f\alpha$: $Z \xrightarrow{\alpha} Z' \xrightarrow{f} X$. We denote this map by $\circ\alpha$:

$$(Z, X) \xleftarrow{\circ\alpha} (Z', X).$$

If we fix X, then sending $Z \rightsquigarrow (Z,X)$ and $\alpha \rightsquigarrow \circ\alpha$ defines a contravariant functor to sets that we denote by $(\,\cdot\,,X)$:

$$\mathcal{C}^\circ \xrightarrow{(\,\cdot\,,X)} (\text{sets}).$$

The symbol $\cdot$ stands for a variable object of the category $\mathcal{C}$.

Similarly, composition on the right with α defines a map

$$(X,Z) \xrightarrow{\alpha\circ} (X,Z')$$

that sends a morphism $X \xrightarrow{g} Z$ to $\alpha \circ g,\ X \xrightarrow{g} Z \xrightarrow{\alpha} Z'$, and this gives us a covariant functor

$$\mathcal{C} \xrightarrow{(X,\,\cdot\,)} (\text{sets})$$

0.6 Morphisms of Functors

morphfunctor

A fact that can be confusing at first is there can be maps between functors. Such a map is called a *morphism*, or *natural transformation* from one functor to another.

Let F and G be two functors $\mathcal{C} \to \mathcal{D}$. A *morphism* $F \xrightarrow{\Phi} G$ consists of morphisms $F(X) \xrightarrow{\Phi(X)} G(X)$ in $\mathcal{D}$, one for each object X of $\mathcal{C}$, such that if $X \xrightarrow{f} Y$ is a morphism of objects of $\mathcal{C}$, the following square of morphisms in $\mathcal{D}$ commutes

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ \downarrow{\scriptstyle\Phi} & & \downarrow{\scriptstyle\Phi} \\ G(X) & \xrightarrow{G(f)} & G(Y) \end{array}$$

For example, on the category of sets, let I be the identity functor and let P be the functor that sends a set X to the product $X \times X$. One can map $I(X)$ to $P(X)$: $X \to X \times X$, by the diagonal map $\Delta(X)$. The diagonal is is a morphism of functors $I \xrightarrow{\Delta} P$.

Or, let $\mathcal{C}$ denote the category of pointed path connected topological spaces. There are two functors from this category to the category of groups, namely π_1 and H_1. Here $\pi_1(X)$ denotes the fundamental group of the space X, and $H_1(X)$ denotes homology with integer coefficients. It is a fact that $H_1(X)$ is the abelianization of $\pi_1(X)$, the group obtained by introducing the commutative law into π_1. Abelianization defines a morphism of functors $\pi_1 \to H_1$.

0.7 Tensor Products

tensprod We have collected facts about tensor products that will be useful here.

Let U and V be modules over a ring R. The *tensor product* $U \otimes_R V$ of U and V is an R-module generated by elements $u \otimes v$ called tensors, with u in U and v in V. Its elements are combinations of tensors with coefficients in R. Since we can absorb a coefficient from R into one of the factors of a tensor, every element of $U \otimes_R V$ can be written as a finite sum $\sum u_i \otimes v_i$.

The module of relations among the tensors is generated by the following *bilinear relations*:

bilinrels (0.7.1) $$(u_1+u_2)\otimes v = u_1 \otimes v + u_2 \otimes v\,, \quad u \otimes (v_1+v_2) = u \otimes v_1 + u \otimes v_2$$

and $$ur \otimes v = u \otimes rv$$

for all u in U, v in V, and r in R. The tensor symbol $\otimes$ is used as a reminder that the elements $u \otimes v$ are manipulated using these relations.

colxrow **0.7.2. Example.** Let U be the space of m dimensional (complex) column vectors, and let V be the space of n-dimensional row vectors. Then $U \otimes_{\mathbb{C}} V$ identifies naturally with the space of $m \times n$-matrices.

There are a few remarks to be made about the definition of tensor product.

(1) The last of the bilinear relations states that scalars move through the tensor symbol. This is why we write the term on the right of that relation as $ur \otimes v$ instead of as $ru \otimes v$. As written, the relations define the tensor product of a *right module* U, a module in which scalars act on the right, and a *left module* V. Since we are working with commutative rings, right modules can be made into left modules simply by setting $ru = ur$. Let's agree that unless stated otherwise, scalar multiplication on the two sides supposed to be equal:

$$ru = ur$$

One can't do this when the ring is noncommutative, and there are situations in which a right R-module is also a left module over a different commutative ring.

(2) The tensor product $U \otimes V$ is made into a left R-module on using the structure of U as left module:

tensormodule (0.7.3)
$$r(u \otimes v) = (ru) \otimes v$$

If there was no left module structure on U, the tensor product wouldn't be a left module.

(3) There is an obvious map $U \times V \xrightarrow{\beta} U \otimes_R V$ from the product *set* to the tensor product that sends (u, v) to $u \otimes v$. However, this map isn't a module homomorphism. As modules, the tensor product $U \otimes_R V$ and the product $U \times V$ aren't closely related. For instance, if U and V are free modules of ranks r and s, then $U \otimes_R V$ is free of rank is rs, while $U \times V$ is free of rank $r + s$.

Instead, the map $U \times V \to U \otimes_R V$ is *bilinear*. It is a universal bilinear map: Any R-bilinear map $U \times V \xrightarrow{f} M$ to a module M can be obtained from a module homomorphism $U \otimes_R V \xrightarrow{\widetilde{f}} M$ by composition, $f = \widetilde{f} \circ \beta$: $\quad U \times V \xrightarrow{\beta} U \otimes_R V \xrightarrow{\widetilde{f}} M$. □

canonisom **0.7.4. Proposition.** *There are canonical isomorphisms*

- $U \otimes_R R \approx U, \quad u \otimes r \leftrightsquigarrow ur$
- $(U \oplus U') \otimes_R V \approx (U \otimes_R V) \oplus (U' \otimes_R V), \quad (u_1 + u_2) \otimes v \leftrightsquigarrow u_1 \otimes v + u_2 \otimes v$

and if R is commutative, then

- $U \otimes_R V \approx V \otimes_R U, \quad u \otimes v \leftrightsquigarrow v \otimes u$
- $(U \otimes_R V) \otimes_R W \approx U \otimes_R (V \otimes_R W), \quad (u \otimes v) \otimes w \leftrightsquigarrow u \otimes (v \otimes w)$

The proofs are very simple. We verify the "distributive law" $(U \otimes_R V) \oplus (U' \otimes_R V) \approx (U \oplus U') \otimes_R V$ as an example. The left side is generated by tensors $u \otimes v$ and $u' \otimes v$, and the relations are the bilinear relations in the two summands. The right side is generated by tensors $x \otimes v$, where $x = u + u'$, with the bilinear relations

$$(x_1 + x_2) \otimes v = x_1 \otimes v + x_2 \otimes v, \quad x \otimes (v_1 + v_2) = x \otimes v_1 + x \otimes v_2, \quad xr \otimes v = x \otimes rv)$$

The relations defining the right side hold on the left side, and setting $x = u + 0$ and $x = 0 + u'$ shows that the relations defining the left side hold in the right side. □

prodbasis **0.7.5. Corollary.** *If U and V are free R-modules with bases $\{u_i\}$ and $\{v_j\}$, respectively, then $U \otimes_R V$ is a free R-module with basis $\{u_i \otimes v_j\}$.* □

rexacttensor **0.7.6. Proposition.** *The tensor product operation is right exact. If*

$$U \xrightarrow{f} U' \xrightarrow{g} U'' \to 0$$

is an exact sequence of R-modules, then for any R-module V, the sequence

$$U \otimes_R V \xrightarrow{f \otimes id} U' \otimes_R V \xrightarrow{g \otimes id} U'' \otimes_R V \to 0$$

is exact.

proof. Let Z be the image of $f \otimes id$, and let $W = (U' \otimes_R V)/Z$. The composed map $(g \otimes id)(f \otimes id)$ is zero, so there is an induced map $W \to U'' \otimes V$. We must show that this map is invertible. To define its inverse, we define a bilinear map $U'' \times V \to W$. Given a tensor $u'' \otimes v$ in $U'' \otimes_R V$, we choose u' in U' such that $g(u') = u''$, and we map $u'' \otimes v$ to the residue of $u' \otimes v$ in W. This is well-defined because if u'_1 and u'_2 are elements of U' such that $g(u'_1) = g(u'_2)$, then $u'_1 - u'_2$ is in the image of f, and $(u'_1 - u'_2) \otimes v$ is in Z. The bilinear relations hold in W because they hold in $U' \otimes_R V$, so this map corresponds to a module homomorphism $U'' \otimes_R V \to W$ that inverts $g \otimes id$. □

tensorrels **0.7.7. Corollary.** **(i)** *Let U, V be R-modules, and let M be an R-moatrix. Suppose that U is presented as R^m/MR^n by an exact sequence*

$$R^n \xrightarrow{M} R^m \to U \to 0$$

Then $U \otimes_R V \approx V^m/MV^n$.
(ii) *With notation as in* **(i)***, suppose that V is presented as R^k/NR^ℓ by an exact sequence*

$$R^\ell \xrightarrow{N} R^k \to V \to 0$$

Then the map $R^m \otimes_R R^k \to U \otimes_R V$ is surjective. Its kernel is generated by the images of the two maps

$$R^n \otimes_R R^k \xrightarrow{M \otimes I} R^m \otimes R^k \quad \text{and} \quad R^m \otimes_R R^\ell \xrightarrow{I \otimes N} R^m \otimes R^k.$$

proof. **(ii)** We form a diagram with exact rows and columns:

$$\begin{array}{ccccccc}
R^n \otimes R^\ell & \longrightarrow & R^m \otimes R^\ell & \longrightarrow & U \otimes R^\ell & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
R^n \otimes R^k & \longrightarrow & R^m \otimes R^k & \longrightarrow & U \otimes R^k & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
R^n \otimes V & \longrightarrow & R^m \otimes V & \longrightarrow & U \otimes V & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}$$

Then the assertion follows from a general fact. For any diagram

$$\begin{array}{ccccccc}
A & \longrightarrow & A' & \longrightarrow & A'' & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
B & \longrightarrow & B' & \longrightarrow & B'' & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
C & \longrightarrow & C' & \longrightarrow & C'' & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}$$

whose rows and columns are exact, the composed map $B' \to C''$ is surjective, and its kernel is the sum of the images of A' and B in B'. The verification is a diagram chase. □

tensornotexact **Example.** This example shows that the tensor product operation isn't exact. Let $R = \mathbb{C}[x]$, and let $\mathbb{C}$ denote the R-module R/xR. When we tensor the exact sequence $0 \to R \xrightarrow{x} R \to \mathbb{C} \to 0$ with $\mathbb{C}$, the result is the non-exact sequence $0 \to \mathbb{C} \xrightarrow{0} \mathbb{C} \to \mathbb{C} \to 0$ □

extendscalars **(0.7.8)** **extension of scalars in a module**

Let $R \xrightarrow{\rho} S$ be a ring homomorphism. An S-module N can be made into an R-module, in which scalar multiplication by an element a of R is defined to be multiplication by its image in S:

restrscalar (0.7.9)
$$ax \overset{def}{=} \rho(a)x$$

This operation is called *restriction of scalars*.

For example, let ρ be the map $\mathbb{C}[t] \to \mathbb{C}$ that evaluates a polynomial p at 0. Restriction of scalars makes a complex vector space V into a $\mathbb{C}[t]$-module in which scalar multiplication is defined by $p(t)v = p(0)v$. This example is trivial, as are all examples of the simple operation of restriction of scalars.

An R, S*-bimodule* is an abelian group that is a left R-module and a right S-module, and such that left and right multiplications commute:

leftrightcommute (0.7.10)
$$r(ms) = (rm)s$$

For example, if we are given a homomorphism $R \to S$, the ring S becomes an R, S-bimodule in which the left operation of R is by restriction of scalars. Then, given a right R-module M, the tensor product $M' = M \otimes_R S$ becomes a right S-module, multiplication by $s \in S$ being $(m \otimes a)s = m \otimes (as)$. This gives a functor
$$R{-}modules \xrightarrow{\otimes S} S{-}modules$$
that is called *extension of scalars*.

locistensor **0.7.11. Corollary.** *Let U and V be modules over a domain R and let s be a nonzero element of R. Let R_s, U_s, V_s be the (simple) localizations of R, U, V, respectively (see 2.8.11).*
(i) *There is a canonical isomorphism $U_s \approx U \otimes_R R_s$.*
(ii) *Localization is compatible with tensor product: $U_s \otimes_{R_s} V_s \approx (U \otimes_R V)_s$* □

fibremodule **(0.7.12) fibres of a module**

Let I be an ideal of a finite-type domain A, and let $\overline{A} = A/I$. Also, let U be an A-module, and let UI be the submodule generated by products $u\alpha$ with u in U and α in I. Tensor product with U givews us a diagram

$$\begin{array}{ccccccccc}
 & & U \otimes_A I & \longrightarrow & U \otimes_A A & \longrightarrow & U \otimes_A \overline{A} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \approx & & \downarrow \approx & & \\
0 & \longrightarrow & UI & \longrightarrow & U & \longrightarrow & U/UI & \longrightarrow & 0
\end{array}$$

in which the left vertical arrow is surjective.

If I is the maximal ideal $\mathfrak{m}_p$ at a point p of $X = \operatorname{Spec} A$, then $\overline{A}$ is the residue field $k(p)$ at p, and $U \otimes_A k(p)$ is the $k(p)$-module obtained from U by extension of scalars. We call this $k(p)$-module the *fibre* of U at p, and we denote it by $U(p)$. Then there is an exact sequence

Utensorktwo (0.7.13)
$$0 \to U\mathfrak{m}_p \to U \to U(p) \to 0$$

that we use to identify the fibre $U(p)$ as $U/U\mathfrak{m}_p = U \otimes_A k(p)$.

The *support* of a finite A-module U is the set of points p such that the fibre $U(p)$ at p isn't zero. The support of a finite module is a closed subset of $X = \operatorname{Spec} A$.

tensoralgebras **(0.7.14) tensor product algebras**

If A and B are algebras over a ring R, the tensor product module $A \otimes_R B$ is made into an R-algebra with multiplication law

$$(\alpha_1 \otimes \beta_1) \cdot (\alpha_2 \otimes \beta_2) = (\alpha_1 \alpha_2) \otimes (\beta_1 \beta_2)$$

and its multiplicative identity is $1 \otimes 1$. One must show compatibility of multiplication with the bilinear relations. Since this is easy, we'll do one verification as an example. We know that $(\alpha_1 + \alpha_1') \otimes \beta_1 = \alpha_1 \otimes \beta_1 + \alpha_1' \otimes \beta_1$, so we must show that

$$\big((\alpha_1 + \alpha_1') \otimes \beta_1\big) \cdot \big(\alpha_2 \otimes \beta_2\big) \;=\; \big(\alpha_1 \otimes \beta_1\big) \cdot \big(\alpha_2 \otimes \beta_2\big) + \big(\alpha_1' \otimes \beta_1\big) \cdot \big(\alpha_2 \otimes \beta_2\big)$$

Using the definition of multiplication, what is to be shown is that $(\alpha_1 + \alpha_1')\alpha_2 \otimes \beta_1\beta_2 = \alpha_1\alpha_2 \otimes \beta_1\beta_2 + \alpha_1'\alpha_2 \otimes \beta_1\beta_2$, which is true.

tensorproperty **0.7.15. Proposition.** *(mapping property of tensor product algebras) Let A, B, and S be R-algebras. Algebra homomorphisms from $A \otimes_R B$ to S correspond bijectively to pairs of algebra homomorphisms from A and B to S:*

$$\mathrm{Hom}_R(A \otimes_R B, S) \approx \mathrm{Hom}_R(A, S) \times \mathrm{Hom}_R(B, S).$$

proof. We note first that sending $\alpha \rightsquigarrow \alpha \otimes 1$ defines an R-algebra homomorphism $A \to A \otimes_R B$. This is pretty clear: $(\alpha + \alpha') \otimes 1 = \alpha \otimes 1 + \alpha' \otimes 1$, $(\alpha\alpha') \otimes 1 = (\alpha \otimes 1)(\alpha' \otimes 1)$, and $(r\alpha) \otimes 1 = r(\alpha \otimes 1)$. Similarly, $\beta \rightsquigarrow 1 \otimes \beta$ defines an R-algebra homomorphism $B \to A \otimes_R B$. This being so, an R-algebra homomorphism $A \otimes_R B \to S$ gives us homomorphisms $A \to S$ and $B \to S$ by composition. Conversely, let algebra homomorphisms $A \xrightarrow{f} S$ and $B \xrightarrow{g} S$ be given. We define $A \otimes_R B \xrightarrow{\varphi} S$ by $\varphi(\alpha \otimes \beta) = f(\alpha)g(\beta)$. To show that φ is well-defined, one must verify the bilinear relations. We verify one as example:

$$\varphi(a \otimes b) + \varphi(a' \otimes b) \;=\; f(a)g(b) + f(a')g(b) \;=\; f(a + a')g(b) \;=\; \varphi((a + a') \otimes b)$$

Then φ is a homomorphism because

$$\varphi(a_1 \otimes b_1)\varphi(a_2 \otimes b_2) \;=\; f(a_1)g(b_1)f(a_2)g(b_2) \;=\; f(a_1a_2)g(b_1b_2) \;=\; \varphi(a_1a_2 \otimes b_1b_2) \qquad \square$$

prodaffvar **(0.7.16) products of affine varieties**

Let $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ be affine varieties, and say that the coordinate rings are presented as $A = \mathbb{C}[x_1, ..., x_m]/(f_1, ..., f_k)$ and $B = \mathbb{C}[y_1, ..., y_n]/(g_1, ..., g_\ell)$. So X and Y are subvarieties of $\mathbb{A}^m$ and $\mathbb{A}^n$, respectively. In the product space $\mathbb{A}^{m+n}$ with coordinates x, y, the product $X \times Y$ of the two varieties is the locus

$$f_1(x) = \cdots = f_k(x) = g_1(y) = \cdots = g_\ell(y) = 0.$$

It is an affine variety whose coordinate ring is $\mathbb{C}[x.y]/(f(x), g(y))$. The algebra $\mathbb{C}[x, y]/\big((f(x), g(y)\big)$ is isomorphic to the tensor product $A \otimes_{\mathbb{C}} B$.

tensorfg **0.7.17. Corollary.** *Let $A = \mathbb{C}[x]/(f)$ and $B = \mathbb{C}[y]/(g)$. Then the product variety* $\operatorname{Spec} A \times \operatorname{Spec} B$ *is isomorphic to the affine variety* $\operatorname{Spec} A \otimes_{\mathbb{C}} B$. $\square$