# Chapter 8 THE RIEMANN-ROCH THEOREM FOR CURVES

curves

version 56



We study smooth projective curves in this chapter. Smooth affine curves were discussed in Chapter 5, and an arbitrary curve is smooth if it has an open covering by smooth affine curves. Our main goal is to prove the Riemann-Roch Theorem, which describes the dimension of the space of rational functions with prescribed poles on a smooth projective curve. Some other facts will be derived from the theorem, among them:

- With its classical topology, a smooth projective curve is a connected manifold.
- The topological genus and the arithmetic genus of a smooth projective curve are equal: $g = p_a$.

## 8.1 Modules on a Curve

curvemodules

Finite modules on a smooth curve have a relatively simple structure, and we make a start at describing them here. Recall that a *torsion element* of a module $M$ over a domain $A$ is an element $m$ such that $am = 0$ for some nonzero element $a$ of $A$, and that a module is *torsion-free* if its only torsion element is 0. These definitions are extended to $\mathcal{O}$-modules by the standard procedure.

tfreelfree

**8.1.1. Proposition.** *A torsion-free $\mathcal{O}$-module $\mathcal{M}$ on a smooth curve $Y$ is locally free: There is an open covering $\{Y^i\}$ of $Y$ such that, for every $i$, the restriction of $\mathcal{M}$ to $Y^i$ is a free module.*

*proof.* A discrete valuation ring $R$ is a principal ideal domain. Every finite $R$-module is a direct sum of its torsion submodule module and a free module, and every finite, torsion-free $R$ module is free. If $Y$ is a smooth curve, its localization at a point $p$ will be a discrete valuation ring. So the localization at $p$ of a torsion-free $\mathcal{O}$-module $\mathcal{M}$ will be free. It follows from the general principle (5.2.4) that $\mathcal{M}$ will be free in a neighborhood of $p$. We'll verify the principle in this case. We replace $Y$ by an affine open neighborhood of $p$, say $Y = \operatorname{Spec} A$. Let $M = \mathcal{M}(Y)$, and let $S$ be the complement of the maximal ideal $\mathfrak{m}$ of $A$ at $p$, so that the localization at $p$ is the valuation ring $R = AS^{-1}$. The localization $MS^{-1}$ of $M$ will be a free module; let $(x_1, ..., x_n)$ be a basis. Then $x_i = s_i^{-1} m_i$, where $s_i$ is an element of $A$ not in the maximal ideal $\mathfrak{m}$ of $p$ and $m_i$ is in $M$. The set $(m_1, ..., m_n)$ is also a basis for $MS^{-1}$. We inspect the homomorphism $A^r \xrightarrow{\varphi} M$ that sends $(a_1, ..., a_n)$ to $\sum a_i m_i$. The kernel $K$ and the cokernel $C$ of $\varphi$ are finite $A$-modules. Since $(m_1, ..., m_n)$ is a basis for $MS^{-1}$, the localized map $R^r \xrightarrow{\varphi'} MS^{-1}$ is an isomorphism. Then, since localization is an exact operation (5.2.6), the localizations $KS^{-1}$ and $CS^{-1}$ are zero. Let $v_1, ..., v_n$ be generators for the finite module $K$. An element $v \in K$ maps to zero in $KS^{-1}$ if it is annihilated by an element $s \in S$. So there is an element $s_i \in S$ such that $s_i v_i = 0$, and by taking a common multiple, we may choose an element $s \in S$ such that $sv_i = 0$ for all $i$. Then the simple localization $K_s$ will be zero. Similarly, there is an element $s \in S$ such that $C_s = 0$, and we may choose an element $s$ so that both $K_s$ and $C_s$ are zero. Then $M_s$ will be free. $\square$

Later in the chapter, we will need the module of homomorphisms from one $\mathcal{O}$-module to another. s(Let $M$ and $N$ be modules over a ring $A$. The set of homomorphisms $M \to N$, which is denoted by

$$\mathrm{Hom}_A(M, N)$$

becomes an $A$-module with some fairly obvious laws of composition: If $\varphi$ and $\psi$ are homomorphisms and $a$ is an element of $A$, then $\varphi + \psi$ and $a\varphi$ are defined by

homismodule (8.1.2) $$[\varphi + \psi](m) = \varphi(m) + \psi(m) \quad \text{and} \quad [a\varphi](m) = a[\varphi(m)] \quad \big(= \varphi(am)\big)$$

homfinite **8.1.3. Lemma.**
**(i)** Hom *is a left exact, contravariant functor of the first variable. An exact sequence* $M_1 \to M_2 \to M_3 \to 0$ *of A-modules produces an exact sequence*

$$0 \to \mathrm{Hom}_A(M_3, N)\, \mathrm{Hom}_A(M_2, N) \to \mathrm{Hom}_A(M_1, N)$$

**(ii)** Hom *is a left exact covariant functor in the second variable. An exact sequence* $0 \to N_1 \to N_1 \to N_3$ *of A-modules produces an exact squence*

$$0 \to \mathrm{Hom}_A(M, N_1) \to \mathrm{Hom}_A(M, N_2) \to \mathrm{Hom}_A(M, N_3)$$

*proof.* **(i)** The map $\mathrm{Hom}_A(M_1, N) \leftarrow \mathrm{Hom}_A(M_2, N)$ is defined to be composition with the map $M_1 \to M_2$. We verify exactness at $\mathrm{Hom}_A(M_2, N)$. A homomorphism $M_2 \xrightarrow{f} N$ such that the composition $M_3 \to M_2 \to N$ is zero induces a map $\overline{f}$ as in the diagram below

$$\begin{array}{ccccccc} M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \overline{f}} & & \\ 0 & \longrightarrow & N & = & N & \longrightarrow & 0 \end{array}$$

$\square$

homfinitetwo **8.1.4. Lemma.** *Let A be a noetherian ring.*
**(i)** *For any finite A-module M, there is an exact sequence* $A^\ell \to A^k \to M \to 0$.
**(ii)** *If $M$ and $N$ are finite A-modules, then* $\mathrm{Hom}_A(M, N)$ *is a finite A-module.*

*proof.* **(i)** A set $m_1, ..., m_k$ that generates the finite module $M$ will define a surjective map $A^k \to M$. Since $A$ is noetherian, the kernel $N$ of this map will be a finite $A$-module. A set $n_1, ..., n_\ell$ that generates $N$ will define a surjective map $A^\ell \to N$. Then the sequence $A^\ell \to A^k \to M \to 0$ will be exact.

**(ii)** We choose a surjection $A^k \xrightarrow{\pi} M$. Lemma 8.1.3 **(i)** tells us that the induced map $\mathrm{Hom}_A(M, N) \to \mathrm{Hom}_A(A^k, N)$ is injective. We note that $\mathrm{Hom}_A(A^k, N) \approx \mathrm{Hom}_A(A, N)^k \approx N^k$. So $\mathrm{Hom}_A(M, N)$ is isomorphic to a submodule of the finite module $N^k$. Since $A$ is noetherian, $\mathrm{Hom}_A(M, N)$ is a finite module. $\square$

The $\mathcal{O}$-module $\underline{\mathrm{Hom}}_{\mathcal{O}}(\mathcal{M}, \mathcal{N})$ of homomorphisms of $\mathcal{O}$-modules on a variety $Y$ is defined using the standard procedure: If $Y' = \operatorname{Spec} A$ is an affine open subset of $Y$, and if $\mathcal{M}(Y') = M$ and $\mathcal{N}(Y') = N$, then the module of sections of $\underline{\mathrm{Hom}}_{\mathcal{O}}(\mathcal{M}, \mathcal{N})$ on $Y'$ is $\mathrm{Hom}_A(M, N)$. The analogues of Lemma 8.1.3 are true for $\underline{\mathrm{Hom}}_{\mathcal{O}}(\mathcal{M}, \mathcal{N})$.

homloc **8.1.5. Proposition.** **(i)** *If $\mathcal{M}$ and $\mathcal{N}$ are finite $\mathcal{O}$-modules on a variety $Y$, $\underline{\mathrm{Hom}}_{\mathcal{O}}(\mathcal{M}, \mathcal{N})$ is also a finite $\mathcal{O}$-module.*
**(ii)** *For any $\mathcal{O}$-module $\mathcal{N}$, $\underline{\mathrm{Hom}}_{\mathcal{O}}(\mathcal{O}, \mathcal{N}) \approx \mathcal{N}$.*

*proof.* **(ii)** This follows from Lemma **??**.

**(i)** The fact that $\underline{\mathrm{Hom}}_{\mathcal{O}}(\mathcal{M},\mathcal{N})$ is a finite module is Lemma 8.1.4 and Proposition 6.3.10.

To verify the coherence property, we must show that if $s$ is a nonzero element of a noetherian domain $A$, there is a canonical isomorphism

$$(\mathrm{Hom}_A(M,N))_s \xrightarrow{\epsilon} \mathrm{Hom}_{A_s}(M_s,N_s)$$

The fact that localization is a functor gives us a module homomorphism $\mathrm{Hom}_A(M,N) \to \mathrm{Hom}_{A_s}(M_s,N_s)$. Since $s$ is invertible in $\mathrm{Hom}_{A_s}(M_s,N_s)$, this homomorphism defines the map $\epsilon$. To show that $\epsilon$ is an isomorphism, we choose a presentation $A^\ell \to A^k \to M \to 0$ as in (8.1.3) **(i)**. Since $\mathrm{Hom}_A(A^r,N) \approx N^r$ for any $r$, it is true that the map $\mathrm{Hom}_A(A^r,N)_s \overset{\to}{\mathrm{Hom}}_{A_s}(A^r_s,N_s)$ is an isomorphism. Then the fact that $\epsilon$ is an isomorphism follows from Lemma 8.1.3 **(i)**.

We omit the proof of the sheaf property. One uses (8.1.4) **(i)** to reduce the problem to the case that $\mathcal{M}=\mathcal{O}$, for which it becomes the sheaf property of $\mathcal{N}$. □ ## give proof??##

dualmodule

### (8.1.6) the dual module

The *rank* of a locally free module $\mathcal{M}$ is the rank of any of its free localizations.

The module $\underline{\mathrm{Hom}}_{\mathcal{O}}(\mathcal{O}^n,\mathcal{O}^m)$ is the free $\mathcal{O}$-module $\mathcal{O}^{m\times n}$ of $m\times n$ matrices. Consequently, if $\mathcal{M}$ and $\mathcal{N}$ are locally free, then $\underline{\mathrm{Hom}}_{\mathcal{O}}(\mathcal{M},\mathcal{N})$ will also be locally free, and if $\mathcal{M}$ and $\mathcal{N}$ have ranks $m$ and $n$, respectively, the rank of $\underline{\mathrm{Hom}}_{\mathcal{O}}(\mathcal{M},\mathcal{N})$ will be $mn$.

The *dual module* $\mathcal{M}^*$ of a locally free $\mathcal{O}$-module $\mathcal{M}$ is the locally free module $\underline{\mathrm{Hom}}_{\mathcal{O}}(\mathcal{M},\mathcal{O})$. The duality between $\mathcal{M}$ and $\mathcal{M}^*$ can be expressed as the $\mathcal{O}$-bilinear form

$$\mathcal{M}^* \times \mathcal{M} \to \mathcal{O}$$

defined by $\langle \varphi, m\rangle = \varphi(m)$, or by the $\mathcal{O}$-module homomorphism

$$\mathcal{M}^* \otimes_{\mathcal{O}} \mathcal{M} \to \mathcal{O}$$

that sends $\varphi\otimes m$ to $\varphi(m)$. The symmetry of these expressions shows that a locally free $\mathcal{O}$-module $\mathcal{M}$ is canonically isomorphic to its double dual $\mathcal{M}^{**}$.

## 8.2 The Modules $\mathcal{O}(D)$

divisors

Let $Y$ be a smooth curve. A $\mathcal{O}$-module that is locally free, of rank one is often called an *invertible* module. An invertible $\mathcal{O}$-module $\mathcal{L}$ is isomorphic to $\mathcal{O}$-module $\mathcal{O}$ in a neighborhood of any point. The term "invertible" is justified by the fact that if $\mathcal{L}$ is invertible, so are $\mathcal{L}^*$ and $\mathcal{L}^*\otimes_{\mathcal{O}}\mathcal{L}$, and the map $\mathcal{L}^*\otimes_{\mathcal{O}}\mathcal{L}\to\mathcal{O}$ will be an isomorphism.

In this section we introduce invertible modules $\mathcal{O}(D)$ associated to divisors $D$ the smooth curve $Y$. These modules are convenient, because they are described simply by divisors. We show that every invertible module is isomorphic to such a module.

A *divisor* on a smooth curve $Y$ is a finite integer combination $r_1p_1+\cdots+r_kp_k$ of points $p_i$ with integer coefficients. A divisor will often be denoted by a capital letter such as $D$. The *degree* of $D=\sum r_ip_i$, the sum $\sum r_i$, will be denoted by $\deg\, D$.

A divisor $D=\sum r_ip_i$ is *effective* if $r_i\geq 0$ for all $i$. If $V$ is an open subset of $Y$, $D$ is *effective on* $V$ if $r_i\geq 0$ for all $p_i$ that are points of $V$.

Let $\mathrm{v}_q$ denote the valuation that corresponds to a point $q$ of $Y$, as before. Recall that a nonzero rational function $f$ is zero at $q$ if $\mathrm{v}_q(f)>0$, and that it has a pole at $q$ if $\mathrm{v}_q(f)<0$. The *divisor* of a rational function $f$ is

$$\mathrm{div}(f)=\sum_q \mathrm{v}_q(f)\, q$$

This divisor is written as a sum over all points of $Y$, but it is actually a finite sum, because $f$ has finitely many zeros and poles. The coefficient $v_q(f)$ will be zero at all other points:

$$\operatorname{div}(f) = \{\text{zeros of } f\} - \{\text{poles of } f\}$$

The divisor of a function is also called a *principal divisor*.

Let $D$ and $E$ be two divisors. By allowing zero coefficients, we may suppose they are combinations of the same finite set $p_1, ..., p_k$ of points, say $D = \sum r_i p_i$ and $E = \sum s_i p_i$. These divisors are *linearly equivalent* if their difference $D - E = \sum (r_i - s_i) p_i$ is a principal divisor.

We now turn to the definition of the $\mathcal{O}$-module associated to a divisor $D$: Roughly speaking, $\mathcal{O}(D)$ is the module of rational functions whose poles are bounded by $D$. More precisely, the sections of $\mathcal{O}(D)$ on an open subset $V$ of $Y$ consist of zero and of the rational functions $f$ such that the the divisor $\operatorname{div}(f) + D$ is effective on $V$. The points of $Y$ that are not in $V$ impose no conditions on the sections of $\mathcal{O}(D)$ on $V$.

Say that $D = \sum r_i p_i$ and that $p_i$ is a point of an open set $V$. If $r_i > 0$, the sections of $\mathcal{O}(D)$ on $V$ have poles of order at most $r_i$ at $p_i$ and if $r_i < 0$, they have zeros of order at least $-r_i$ at $p_i$.

For example, suppose that $D = p$. If an open set $V$ contains $p$, the nonzero sections of $\mathcal{O}(p)$ on $V$ are the rational functions with a pole of order at most 1 at $p$, that are regular at every other point of $V$, and that can have arbitrary zeros or poles at points not in $V$. If $V$ does't contain $p$, the sections of $\mathcal{O}(p)$ on $V$ are the regular functions on $V$. The module $\mathcal{O}(-p)$ is the maximal ideal at $p$. If an open set $V$ contains $p$, the sections of $\mathcal{O}(-p)$ on $V$ are the regular functions on $V$ that vanish at $p$, and if $p$ doesn't contain $V$, the sections of $\mathcal{O}(-p)$ on $V$ are the regular functions on $V$.

The fact that a section of $\mathcal{O}(D)$ is allowed to have a pole at $p_i$ if its coefficient $r_i$ is positive contrasts with the divisor of a function. If $\operatorname{div}(f) = \sum a_i p_i$, then $a_i > 0$ means that $f$ has a zero at $p_i$. Thus, if $D = \operatorname{div}(f)$, $f$ will be a global section of $\mathcal{O}(-D)$.

Let $D$ and $E$ be divisors. It is obvious that $\mathcal{O}(D)$ is a submodule of $\mathcal{O}(E)$ if and only if $E - D$ is effective. But when regarded simply as $\mathcal{O}$-modules, there will be homomorphisms between them that aren't simple inclusions. Thus homomorphisms $\mathcal{O} \xrightarrow{f} \mathcal{O}(D)$ are given as multiplication by global sections $f$ of $\mathcal{O}(D)$. (The analogous statement is true for global sections of any $\mathcal{O}$-module. See Lemma 6.4.2.) For instance, regular functions $f$ on $Y$ correspond bijectively to module homomorphisms $\mathcal{O} \xrightarrow{f} \mathcal{O}$.

The next proposition describes nonzero ideals in the structure sheaf of a of a smooth curve $Y$ as modules of the form $\mathcal{O}(-D)$. It extends the description of closed subsets of $\mathbb{P}^1$ given in (**??**).

idealincurve **8.2.1. Proposition.** *Let $\mathcal{I}$ be a nonzero ideal in the structure sheaf $\mathcal{O}$ of a smooth curve $Y$.*
**(i)** *There are points $p_1, ..., p_k$ of $Y$ and integers $r_1, ..., r_k$ such that $\mathcal{I}$ is the product $\mathfrak{m}_1^{r_1} \cdots \mathfrak{m}_k^{r_k}$ of the maximal ideals $\mathfrak{m}_i$ at $p_i$.*
**(ii)** *Let $p_i$ and $r_i$ be as in* **(i)***, and let $D$ denote the effective divisor $\sum r_i p_i$. Then $\mathcal{I} = \mathcal{O}(-D)$.*

*proof.* **(i)** Because $Y$ has dimension one, the support of the quotient algebra $\mathcal{O}/\mathcal{I}$ is a finite set $\{p_1, ..., p_k\}$ of points of $Y$. To show that $\mathcal{I} = \mathfrak{m}_1^{r_1} \cdots \mathfrak{m}_k^{r_k}$, it is enough to show that there is an open covering on which those ideals are equal. Therefore it suffices to prove the assertion for an affine open set $Y' = \operatorname{Spec} A$ that contains just one of the points $p = p_i$, with maximal ideal $\mathfrak{m}$. Then it follows from Proposition 5.4.5.

**(ii)** This follows from the definition of $\mathcal{O}(-D)$. □

Various facts about these modules $\mathcal{O}(D)$ are collected below. The proofs are simple enogh that we omit them.

OtoODone **8.2.2. Proposition.** **(i)** *For any divisor $D$, $\mathcal{O}(D)$ is an invertible $\mathcal{O}$-module.*

**(ii)** *If $D$ and $E$ are divisors, $\mathcal{O}(D+E)$ is canonically isomorphic to $\mathcal{O}(D) \otimes_{\mathcal{O}} \mathcal{O}(E)$.*

**(iii)** *Let $E_1$ and $E_2$ be divisors. The modules $\mathcal{O}(E_1)$ and $\mathcal{O}(E_2)$ are isomorphic if and only if the divisors are linearly equivalent. If $E_1 - E_2 = \operatorname{div}(f)$, multiplication by $f$ defines an isomorphism $\mathcal{O}(E_1) \xrightarrow{f} \mathcal{O}(E_2)$. In particular, if $\operatorname{div}(f) = D$ multiplication by $f$ defines an isomorphism $\mathcal{O}(D) \longrightarrow \mathcal{O}$.*

zerospoles **(8.2.3)** **sections of invertible modules**

Let $\mathcal{L}$ be an invertible $\mathcal{O}$-module on a smooth curve $Y$, and let $K$ be the function field of $Y$. Because $\mathcal{L}$ is locally isomorphic to $\mathcal{O}$, $\mathcal{L}_K = \mathcal{L} \otimes_{\mathcal{O}} K$ will be a one-dimensional $K$-vector space. A nonzero element of $\mathcal{L}_K$ is called a *rational section* of $\mathcal{L}$. A rational section will be a section on any sufficiently small (nonempty) open subset $V$. If $s$ is one rational section of $\mathcal{L}$, then since $\mathcal{L}_K$ is one-dimensional, every rational section has the form $s' = fs$, where $f$ is a rational function.

Let $s$ be a rational section of $\mathcal{L}$, let $q$ be a point of $Y$, and and let $V$ be an open neighborhood of a point $q$ on which there is an isomorphism $\mathcal{L} \xrightarrow{\varphi} \mathcal{O}$. One says that $s$ has a *zero of order* $r$ at $q$ if the function $f = \varphi(s)$ that corresponds to $s$ has a zero of order $r$ at $q$. If $\mathcal{L} \xrightarrow{\psi} \mathcal{O}$ is another local isomorphism, the map $\varphi\psi^{-1}$ will be a local isomorphism $\mathcal{O} \to \mathcal{O}$. It will be multiplication by an invertible function. Therefore the order of zero of $f = \varphi(s)$ will be equal to that of $g = \psi(s)$.

Similarly, a rational section $s$ of $\mathcal{L}$ has a pole of order $r$ at $q$ if the corresponding rational function $f = \varphi(s)$ has a pole of that order at $q$.

Therefore the *divisor* $\text{div}(s)$ of a rational section $s$ is defined. A rational section $s$ will be a section of $\mathcal{L}$ on an open set $V$ if and only if $\text{div}(s)$ is effective on $V$.

This allows us to define modules $\mathcal{L}(D)$ analogous to the modules $\mathcal{O}(D)$. Given a divisor $D$, the sections of $\mathcal{L}(D)$ on an open set $V$ are the rational sections $s$ such that $\text{div}(s) + D$ is effective on $V$, together with zero. Then if $\mathcal{L} \subset \mathcal{L}'$ are invertible modules, then $\mathcal{L}' = \mathcal{L}(D)$ for some effective divisor $D$.

##cut this?##

If a section $s$ of $\mathcal{L}$ on an open set $V$ is nonzero, the homomorphism $\mathcal{O} \xrightarrow{s} \mathcal{L}$ of multiplication by $s$, which is defined on $V$, will be injective simply because $\mathcal{L}$ is torsion-free. And if $s$ doesn't vanish at any point of $V$, the corresponding function $f = \varphi(s)$ won't vanish either. Then multiplication by $f$ will be a bijection $\mathcal{O} \to \mathcal{O}$, and therefore multiplication by $s$ will be a bijection $\mathcal{O} \to \mathcal{L}$ too. In particular, the invertible module $\mathcal{L}$ is isomorphic to $\mathcal{O}$ if and only if it has a nonvanishing global section. A bijective map $\mathcal{O} \to \mathcal{L}$ is given by a global section $s$ of $\mathcal{L}$ that is nowhere zero.

OtoODtwo **8.2.4. Proposition.** *Every invertible $\mathcal{O}$-module $\mathcal{L}$ is isomorphic to a module of the form $\mathcal{O}(D)$.*

For example, suppose that $Y$ is a closed subvariety of projective space $\mathbb{P}^n$. Then by definition, $\mathcal{O}_Y(1)$ is the invertible module $\mathcal{O}_Y \otimes \mathcal{O}_{\mathbb{P}^n}(1)$. The global section $x_0$ of $\mathcal{O}_{\mathbb{P}^n}(1)$ restricts to a global section of $\mathcal{O}_Y(1)$. Let's denote this restriction by $\overline{x}_0$. The divisor $D = \text{div}(\overline{x}_0)$ is the divisor of zeros of $x_0$ in $Y$. Thus $\mathcal{O}_Y(1) \approx \mathcal{O}_Y(D)$.

*proof.* Let $s$ be a (nonzero) rational section of $\mathcal{L}$, and let $D$ be its divisor. Then $s$ is a global section of $\mathcal{L}(-D)$ that isn't zero at any point of $Y$. This global section defines an isomorphism $\mathcal{O} \xrightarrow{s} \mathcal{L}(-D)$. Thus cl $\approx \mathcal{O}(D)$. $\square$

## 8.3 Cohomology

cohcurve Let $Y$ be a smooth projective curve. A we saw in Chapter 7, the cohomology $H^q(Y, \mathcal{M})$ of an $\mathcal{O}$-module $\mathcal{M}$ is zero when $q > 1$ and if $\mathcal{M}$ is a finite module, then $H^0(Y, \mathcal{M})$ and $H^1(Y, \mathcal{M})$ are finite dimensional vector spaces. The *Euler characteristic* of a finite $\mathcal{O}$-module $\mathcal{M}$ is

chicurve (8.3.1)
$$\chi(\mathcal{M}) = \dim H^0(Y, \mathcal{M}) - \dim H^1(Y, \mathcal{M}).$$

In particular,

$$\chi(\mathcal{O}_Y) \;=\; \dim H^0(Y, \mathcal{O}_Y) - \dim H^1(Y, \mathcal{O}_Y)$$

We will see below that $H^0(Y, \mathcal{O}_Y) = \mathbb{C}$, and by definition, $\dim H^1(Y, \mathcal{O}_Y) = p_a$ is the *arithmetic genus* of $Y$. Thus

$$\chi(\mathcal{O}_Y) \;=\; 1 - p_a$$

To determine the effect on the cohomology of an $\mathcal{O}$-module $\mathcal{O}(D)$ when we allow one more zero or pole, we consider the inclusion map $\mathcal{O}(D-p) \xrightarrow{\varphi} \mathcal{O}(D)$ and form a short exact sequence

$$0 \to \mathcal{O}(D-p) \to \mathcal{O}(D) \to \epsilon \to 0$$

where $\epsilon$ is the cokernel of $\varphi$. Recall that $\mathcal{O}(-p) = \mathfrak{m}_p$. So this sequence above can be obtained from the sequence

$$0 \to \mathcal{O}(-p) \to \mathcal{O} \to \kappa_p \to 0$$

by tensoring with $\mathcal{O}(D)$. Since $\mathcal{O}(D)$ is locally isomorphic to $\mathcal{O}$, $\epsilon = \kappa_p \otimes_{\mathcal{O}} \mathcal{O}(D)$ is a one-dimensional vector space supported on the point $p$. Therefore $H^0(X, \epsilon)$ has dimension 1 and $H^1(X, \epsilon) = 0$.

Let's abbreviate by writing [1] for the one-dimensional space $H^0(Y, \epsilon)$. The cohomology sequence of the sequence above becomes

addpoint (8.3.2) $$0 \to H^0(Y, \mathcal{O}(D-p)) \to H^0(Y, \mathcal{O}(D)) \to [1] \to H^1(Y, \mathcal{O}(D-p)) \to H^1(Y, \mathcal{O}(D)) \to 0$$

Consequently, when we change a divisor by adding a point, one of two things happens: Either the dimension of $H^0$ increases by 1, or the dimension of $H^1$ decreases by 1. In either case,

chichange (8.3.3) $$\chi(\mathcal{O}(D)) = \chi(\mathcal{O}(D-p)) + 1.$$

Moreover, when we add an infinite sequence of points in succession, then because $H^1$ is finite-dimensional, its dimension can decrease only finitely often. Therefore $H^0$ will tend to infinity.

RRcurve **8.3.4. Riemann-Roch Theorem (version 1).** *Let $D = \sum r_i p_i$ be a divisor on a smooth projective curve $Y$. Then*

$$\chi(\mathcal{O}(D)) \quad = \quad \chi(\mathcal{O}) + \deg\, D$$

The theorem follows from (8.3.3), because we can get from $\mathcal{O}$ to $\mathcal{O}(D)$ by a finite number of operations, each of which changes the divisor by adding or subtracting a point. □

The full Riemann-Roch Theorem is more precise than this version. It describes the Euler characteristic of a locally free $\mathcal{O}$-module $\mathcal{M}$, and it identifies the cohomology $H^1(Y, \mathcal{M})$ as the dual space to the space of global sections of another locally free $\mathcal{O}$-module $\mathcal{M}^D$, the *Serre dual* of $\mathcal{M}$ (see Section 8.7 below). However, the weaker version 1 has important consequences.

RRcor **8.3.5. Corollaries.** *Let $Y$ be a smooth projective curve.*

**(i)** *The divisor* $\operatorname{div}(f)$ *of a nonzero rational function $f$ has degree zero: The number of zeros of $f$ is equal to the number of its poles. Therefore linearly equivalent divisors have equal degrees.*

**(ii)** *A nonconstant rational function $f$ on $Y$ takes every value, including infinity, the same number of times.*

**(iii** *A rational function that is regular at every point of $Y$ is a constant: $H^0(Y, \mathcal{O}) = \mathbb{C}$.*

**(iv)** *If $D$ is a divisor on $Y$, then* $\dim H^0(Y, \mathcal{O}(D)) \geq \deg\, D + 1 - p_a$.

**(v)** *If* $\deg\, D \geq p_a$*, then* $H^0(Y, \mathcal{O}(D)) \neq 0$.

**(vi)** *If* $\deg\, D < 0$*, then* $H^0(Y, \mathcal{O}(D)) = 0$.

*proof.* **(i)** Let $D = \operatorname{div}(f)$. Multiplication by $f$ defines an isomorphism $\mathcal{O}(D) \to \mathcal{O}$ (8.2.2), so $\chi(\mathcal{O}(D)) = \chi(\mathcal{O})$. On the other hand, by Riemann-Roch, $\chi(\mathcal{O}(D)) = \chi(\mathcal{O}) + \deg\, D$. Therefore $\deg\, D = 0$.

**(ii)** For any complex number $c$, the functions $f$ and $f - c$ have the same poles. Therefore the number of zeros of $f - c$, which is the number of points at which $f$ takes the value $c$, is equal to the number of poles of $f$.

**(iii)** According to **(ii)**, a nonconstant function must have a pole.

**(iv),(v)** follow directly from **(iii)** and the Riemann-Roch formula.

**(v)** This follows from **(ii)** because, when $\deg\, D < 0$, a global section of $\mathcal{O}(D)$ would have more zeros than poles. □

Corollary 8.3.5 **(iii)** and Proposition 8.2.2 **(iii)** allow us to define the *degree* $\deg \mathcal{L}$ of an invertible $\mathcal{O}$-module $\mathcal{L}$. It is the degree of a divisor $D$ such that $\mathcal{L} \approx \mathcal{O}(D)$. With this definition of degree, Riemann-Roch for $\mathcal{L}$ becomes

RRL (8.3.6)
$$\chi(\mathcal{L}) = \chi(\mathcal{O}) + \deg \mathcal{L}$$

curveconn **8.3.7. Theorem.** *With its classical topology, a smooth projective curve $Y$ is connected.*

The proof that $Y$ is a compact orientable manifold can be adapted from the case of plane curves (see **??**).

*proof.* This is a proof by contradiction. Suppose that the curve is the union of disjoint closed subsets: $Y = Y_1 \cup Y_2$. Both $Y_1$ and $Y_2$ will be compact manifolds. We choose a point $p$ of $Y_1$. Riemann-Roch shows that $H^0(Y, \mathcal{O}(np))$ isn't zero, if $n$ is sufficiently large. So there will be a nonconstant rational function $f$ that is regular on the complement of $p$. Then $f$ will have no pole on $Y_2$. It will be a bounded analytic function on that compact space. The maximum principle for analytic functions shows that $f$ will be constant on $Y_2$. We may subtract this constant, to construct a nonzero rational function that is identically zero on $Y_2$. Similarly, there is a nonzero rational function $g$ that is identically zero on $Y_1$. Then $fg = 0$. This is impossible because the rational functions form a field. □

## 8.4 Curves as Coverings of $\mathbb{P}^1$, again

covercurve We will want to view a smooth projective curve $Y$ as a covering of the projective line $X = \mathbb{P}^1$, as we did with plane curves in the first chapter. If $Y$ is a closed subvariety of $\mathbb{P}^n$, we can construct a morphism $Y \xrightarrow{\pi} X$ by restriction from the projection $\mathbb{P}^n \to \mathbb{P}^1$ that sends $(x_0, ..., x_n)$ to $(x_0, x_1)$. The center of projection, the locus where the projection isn't defined, is the linear subspace $L$ of points $(0, 0, x_2, ..., x_n)$. It is defined by two equations $x_0 = x_1 = 0$. Provided that coordinates are in general position, $Y$ will meet the hyperplane $\{x_0 = 0\}$ in a finite number of points, none of which are in the locus $\{x_1 = 0\}$. Then $Y \cap L$ will be empty, and $\pi$ will be a finite morphism. This follows from Chevalley's Finiteness Theorem 4.5.2. So $\pi$ will present $Y$ as a branched covering of $X$. One could also construct $Y$ as the normalization of $X$ in the function field $L$ of $Y$. The direct image $\pi_*\mathcal{O}_Y$ will be a finite torsion-free, and therefore locally free, $\mathcal{O}_X$-module. Its rank as $\mathcal{O}_X$-module is the *degree* of the covering. Let's denote the degree by $n$.

The inverse image $\pi^{-1}X'$ of an affine open subset $X' = \operatorname{Spec} A'$ of $X$ will be a smooth affine curve $Y' = \operatorname{Spec} B'$, and $B'$ will be a finite, locally free $A'$-algebra whose rank is the degree $n$ of $Y$ over $X$.

The local analytic structure of the covering $Y/X$ is very simple, and because it is useful for intuition, we describe it here. Let $q$ be a point of $Y$ that maps to a point $p$ of $X$. We arrange coordinates so that $p$ is the point $x = 0$ of $X$. Let $\mathrm{v}_q$ be the valuation associated to $q$, and let $z$ be an element of the local ring $R$ at $q$ that generates its maximal ideal, an element with $\mathrm{v}_q(z) = 1$. Then $z$ will be an analytic coordinate function, locally at $q$. The order of vanishing $e = \mathrm{v}_q(x)$ of $x$ at $q$ is the *ramification index* of the covering at $q$. Then in $R$, we will have $x = uz^e$, where $u$ is a unit. We normalize the value $u(q)$ of $u$ to 1. Working analytically, the function $u$ has a unique $e$th root $w$ with $w(q) = 1$. The function $y = wz$ is a local analytic coordinate at $q$, and $x = (wz)^e = y^e$.

yethroot **8.4.1. Corollary. (i)** *Analytically, any covering $Y \to X$ of smooth curves is locally isomorphic to the $e$th root covering: $y^e = x$.*

**(ii)** *Let $e$ be the ramification index of the covering at a point $q$, and let $p$ be the image of $q$. As a point $p'$ of $X$ approaches $p$, $e$ points of the fibre over $p'$ approach $q$.*

**(iii)** *Let $q_1, ..., q_k$ be the points of $Y$ that lie over a point $p$ of $X$ over $p$, and let $\mathrm{v}_i(x) = e_i$ be the ramification index at $q_i$. Then the degree $n$ of $Y$ over $X$ is $\sum e_i$.* □

workonX **8.4.2. Notation.** When considering a smooth projective curve $Y$ as a covering of $X = \mathbb{P}^1$, it will be convenient to work primarily on $X$, and we will often pass between an $\mathcal{O}_Y$-module $\mathcal{M}$ and its direct image $\pi_*\mathcal{M}$. Recall that if $X'$ is open in $X$, then $[\pi_*\mathcal{M}](X') = \mathcal{M}(Y')$, where $Y' = \pi^{-1}X'$. If we restrict the $\mathcal{O}$-module by looking only at open subsets of $Y$ of the form $\pi^{-1}X'$, the only difference between $\mathcal{M}$ and its direct image will be the position of the symbol $\pi$ in the notation: $[\pi_*\mathcal{M}](X') = \mathcal{M}(\pi^{-1}X')$.

So one can think of the direct image $\pi_*\mathcal{M}$ as working with $\mathcal{M}$, but looking only at open subsets of $Y$ that are inverse images of open subsets of $X$. To simplify notation, we omit the symbol $\pi_*$, and simply write $\mathcal{M}$ for $\pi_*\mathcal{M}$. If $X'$ is open in $X$, $\mathcal{M}(X')$ will stand for $\mathcal{M}(\pi^{-1}X')$. To eliminate confusion, we may refer to an $\mathcal{O}_Y$-module $\mathcal{M}$ as an $\mathcal{O}_X$-module when thinking of its direct image. Because $H^q(X,\pi_*\mathcal{M}) = H^q(Y,\mathcal{M})$ (7.4.18), dropping the symbol $\pi_*$ won't get us into trouble with cohomology.

In accordance with this convention, we will also write $\mathcal{O}_Y$ for $\pi_*\mathcal{O}_Y$. In order not to confuse $\mathcal{O}_Y$ with $\mathcal{O}_X$, we must be careful to include the subscripts.

## 8.5 Differentials

diff

Exactly why differentials enter into the Riemann-Roch Theorem is something of a mystery, but they do, so we introduce them here.

Let $M$ be a module over an algebra $A$. A *derivation* $A \xrightarrow{\delta} M$ is a $\mathbb{C}$-linear map such that

deriv (8.5.1)
$$\delta(ab) = a\,\delta b + b\,\delta a$$

for all $a,b$ in $A$, and $\delta c = 0$ for all $c$ in $\mathbb{C}$. For example, let $A$ be the polynomial ring $\mathbb{C}[x_1,...,x_n]$, and let $m_1,...,m_n$ be elements of an $A$-module $M$. The map $A \xrightarrow{\delta} M$ defined by $\delta(f) = \sum \frac{\partial f}{\partial x_i} m_i$ is a derivation.

The *module of differentials* $\Omega_A$ of the algebra $A$, also called *Kähler differentials.* is an $A$-module generated by elements denoted by $da$, one for each $a$ in $A$, with the relations that make the map $A \xrightarrow{d} \Omega_A$ that sends $a$ to $da$ a derivation:

$$d(ab) = a\,db + b\,da \;\text{ for }\; a,b \;\text{ in }\; A, \text{ and }\; dc = 0 \text{ for } c \text{ in } \mathbb{C}$$

These relations show by induction that $dx^k = kx^{k-1}dx$, and that if $f(x)$ is a polynomial, then $df = \frac{df}{dx}dx$.

omegafree **8.5.2. Proposition.** *Let $R$ denote the polynomial ring $\mathbb{C}[x_1,...,x_n]$. The $R$-module $\Omega_R$ of differentials is a free $R$-module with basis $dx_1,...,dx_n$.*

*proof.* The formula $df = \sum \frac{\partial f}{\partial x_i} dx_i$ shows that the elements $dx_1,...,dx_n$ generate $\Omega_R$ as $R$-module. If $F$ denotes the free $R$-module with basis $v_1,...,v_n$, then sending $f \rightsquigarrow \sum \frac{\partial f}{\partial x_i} v_i$ is a derivation that induces a surjective module homomorphism $\Omega_R \xrightarrow{\varphi} F$. Since $F$ is free, $\varphi$ is an isomorphism. □

homomderiv **8.5.3. Lemma. (i)** *Composition with the derivation $A \xrightarrow{d} \Omega_A$ defines a bijection between homomorphisms of $A$-modules $\Omega_A \xrightarrow{\varphi} M$ and derivations $A \xrightarrow{\delta} M$.*

**(ii)** *An algebra homomorphism $A \xrightarrow{\varphi} B$ induces a module homomorphism $\Omega_A \xrightarrow{d\varphi} \Omega_B$ making a diagram*

$$\begin{array}{ccc} B & \xrightarrow{d} & \Omega_B \\ {\scriptstyle\varphi}\uparrow & & \uparrow{\scriptstyle d\varphi} \\ A & \xrightarrow{d} & \Omega_A \end{array}$$

*commutes.*

*proof.* **(i)** This follows from the defining relations for $\Omega_A$.

**(ii)** This follows from the mapping property **(i)**, because the composed map $A \xrightarrow{\varphi} B \xrightarrow{d} \Omega_B$ will be a derivation. □

**8.5.4. Lemma. (i)** *Let $I$ be an ideal of an algebra $R$, and let $A = R/I$. Suppose that $I$ is generated by*
omegafreetwo *elements $f = (f_1,...,f_r)$ of $R$. Then $\Omega_A$ is the quotient of $\Omega_R$ obtained from $\Omega_R$ by introducing the two rules:*

- *multiplication by $f_i$ is zero, and*
- *$df_i = 0$*

**(ii)** *Let $S$ be a multiplicative system in a domain $A$. There is a canonical isomorphism $\Omega_A S^{-1} \to \Omega_{AS^{-1}}$. In particular, if $K$ is the field of fractions of $A$, then $\Omega_K \approx \Omega_A \otimes_A K$.*

*proof.* The statment can be summed up by an exact sequence

omegase-quence (8.5.5)
$$I \xrightarrow{d} \Omega_R \otimes_R A \to \Omega_A \to 0$$

in which the map labeled $d$ sends an element $g$ of $I$ to $dg \otimes 1$.

The explanation of this sequence is this: $\Omega_R \otimes_R A \approx \Omega_R / I\Omega_R$ is the result of declaring that multiplication by $f_i$ is zero. Then the assertion becomes that killing the image $dI$ of $I$ in $\Omega_R \otimes_R A$ produces $\Omega_A$. To see this, let $S$ denote the module obtained from $\Omega_R \otimes_R A$ by killing $dI$. We note first that multiplication by $f_i$ and $df_i$ must be zero in $\Omega_A$. Therefore there is a canonical module homomorphism $S \to \omega_A$. Conversely the composed map $R \xrightarrow{d} \Omega_R \to S$ is a derivation, and $I$ is in its kernel. Let's denote that derivation by $\overline{d}$. We define a derivation $A \xrightarrow{\delta} S$ as follows: Given $a \in A$, we represent $a$ by an element $r \in R$, and we define $\delta(a) = \overline{d}(r)$. This is well-defined because, if $r \in I$, then because $dI = 0$, $\overline{d}(r) = 0$. Then $\delta$ is a derivation because $\overline{d}$ is one. By the universal property of $\Omega_A$, the map $S \to \Omega_A$ is invertible.

**(ii)** The composition $A \to AS^{-1} \to \Omega_{AS^{-1}}$ is a derivation. It defines an $A$-module map $\Omega_A \to \Omega_{AS^{-1}}$, and therefore a map $\Omega_A S^{-1} \to \Omega_{AS^{-1}}$. To invert this map, we define a derivation $AS^{-1} \xrightarrow{\delta} \Omega_A S^{-1}$. Setting $d(s^{-1}) = -s^{-2}ds$ for $s$ in $S$, we define $\delta$ by

$$\delta(as^{-1}) = (ds^{-1})a + s^{-1}da = -as^{-2}ds + s^{-1}da)$$

and we check that this is a derivation. □

When $X$ is a variety, the $\mathcal{O}$-module $\Omega_X$ of differentials on $X$ is defined using the standard procedure: If $X' = \operatorname{Spec} A$ is an affine open, the $A$-module of sections of $\Omega_X$ on $X'$ is $\Omega_A$.

omegafunct **8.5.6. Proposition.** *The $\mathcal{O}_Y$-module $\Omega_Y$ of differentials on a smooth curve $Y$ is an invertible module.*

*proof.* The coherence property, the sheaf property, and the fact that $\Omega_Y$ is a finite module, all follow from Proposition 8.5.4.

To show that the module is invertible, we may assume that $Y$ is affine, say $Y = \operatorname{Spec} B$. Let $q$ be a point of $Y$, and let $y$ be an element with $\mathrm{v}_q(y) = 1$, a local generator for the maximal ideal $\mathfrak{m}_q$. We show that $dy$ generates $\Omega_B$ locally. We are allowed to replace $B$ by any simple localization $B_s$, provided that $s$ is not zero at $q$. To begin with, we localize so that $y$ generates $\mathfrak{m}_q$.

Next, we recall that if $\mathfrak{m}$ is the maximal ideal of $B$ at a point $q$, the quotient ring $\overline{B} = B/\mathfrak{m}^n B$ is a truncated polynomial ring $\overline{B} = \mathbb{C}[t]/(t^n)$, where $t$ is a local generator for $\mathfrak{m}$ (see Proposition 5.4.5 **(iii)**).

*Step 1:* Applying the Nakayama Lemma.

We want to show that, when $B$ is replaced by suitable localization, every differential $df$ with $f$ in $B$ will be a multiple of $dy$. Since $B/\mathfrak{m}^2 B$ is a truncated polynomial ring. we can write $f$ in the form $f = c_0 + c_1 y + y^2 h$ for some $h$ in $B$. Then

$$df = c_1 dy + (y^2 dh + 2y, dy) = c_1 dy + y\beta$$

with $\beta$ in $\Omega_B$. Therefore

$$\Omega_B = B\,dy + \mathfrak{m}\Omega_B$$

If $\overline{\Omega}$ denotes the quotient module $\Omega_B / B\,dy$, we have $\overline{\Omega} = \mathfrak{m}\overline{\Omega}$. The Nakayama Lemma tells us that there is an element $z \in \mathfrak{m}$ such that $s = 1 - z$ annihilates $\overline{\Omega}$. When we replace $B$ by the localization $B_s$, we will have $\overline{\Omega} = 0$ and $\Omega_B = B\,dy$, as required.

*Step 2:* Completion of the proof.

We now know that, after a suitable localization, $\Omega_B$ will be generated by an element $dy$. Is $\Omega_B$ then a free $B$-module? We must still verify that $dy$ isn't a torsion element. Suppose that $s$ is a nonzero element of $B$ and that $s\,dy = 0$. Then, when we replace $B$ by $B_s$, we will have $\Omega_B = 0$. Let $\mathfrak{m}$ be a maximal ideal of this localized $B$, and let $\overline{B} = B/\mathfrak{m}^n B$ with $n > 1$. The surjective map $B \to \overline{B}$ gives us a surjection $\Omega_B \to \Omega_{\overline{B}}$ (Lemma 8.5.4). Since $\overline{B}$ is a truncated polynomial ring $\mathbb{C}[t]/(t^n)$, $\Omega_{\overline{B}}$ is the $\overline{B}$-module generated by $dt$, with the relation $t^{n-1}dt = 0$ (Lemma 8.5.4 again). It is not the zero module, and therefore $\Omega_B$ isn't zero either. □

canonical-divisor **8.5.7. Corollary.** *There is a divisor $K$, determined up to linear equivalence such that $\Omega_Y = \mathcal{O}(K)$.*

This follows from Propositions 8.5.6, 8.2.2, and 8.2.4. □

The divisor $K$ mentioned in the corollary is called a *canonical divisor*. It is convenient to represent $\Omega_Y$ as a module $\mathcal{O}(K)$, though the canonical divisor is determined only up to linear equivalence.

## 8.6 Trace

tracediff In this section and the next, we present an adaptation of Grothendieck's proof of duality to Riemann-Roch for curves. The heart of the proof is a subtle fact, that there is a trace map for differentials analogous to the trace of a function.

tracefn **(8.6.1) trace of a function, revisited**

Let $Y \xrightarrow{\pi} X$ be a finite morphism from a smooth projective curve $Y$ to $X = \mathbb{P}^1$ as before, and let $F$ and $K$ be the function fields of $X$ and $Y$, respectively. So $K$ is a finite field extension of the rational function field $F = \mathbb{C}(x)$, and the degree of the field extension is equal to the degree $n$ of $Y$ over $X$.

We denote the *trace map* for functions, which carries $K$ to $F$, by $\mathrm{tr}$. The trace of an element $g$ of $K$ was defined before (4.3.6), as the trace of the operator of multiplication by $g$ on the $F$-vector space $K$. The trace carries regular functions to regular functions: If $X' = \operatorname{Spec} A'$ is an affine open subset of $X$ whose inverse image is $Y' = \operatorname{Spec} B'$, then because $A'$ is normal, the trace of an element of $B'$ will be in $A'$ (4.3.4).

The trace can be described analytically as a sum over the sheets of the covering $Y \to X$: Let $p$ be a point of $X$ that is not a branch point for the covering. There will be $n$ points $q_1, ..., q_n$ in the fibre over $p$. If $U$ is a small neighborhood of $p$ in the classical topology, its inverse image $V = \pi^{-1}U$ will consist of disjoint neighborhoods $V_i$ of $q_i$, each of which maps bijectively to $U$, and the ring of analytic functions on $V_i$ will be isomorphic to the ring $R$ of analytic functions on $U$. This allows us to identify the ring of analytic functions on $V$ with the direct sum of $n$ copies of $R$. If a rational function $g$ on $Y$ is regular on $V$, its restriction to $V$ can be written as $g = g_1 \oplus \cdots \oplus g_n$, with $g_i$ in $R$. The matrix of left multiplication by $g$ on $R \times \cdots \times R$ is the diagonal matrix with entries $g_i$. Therefore the trace of $g$ is

trsum (8.6.2)
$$\mathrm{tr}(g) = g_1 + \cdots + g_n$$

tracesum **8.6.3. Lemma.** *With $Y \xrightarrow{\pi} X$ as above, let $p$ be a point of $X$, possibly a branch point, and let $q_1, ..., q_k$ be the fibre over $p$. Let $e_i$ be the ramification index at $q_i$. If $g$ is a rational function on $Y$ that is regular at the points $q_1, ..., q_k$, the trace $\mathrm{tr}(g)$ is regular at $p$, and its value there is*

$$[\mathrm{tr}(g)](p) = \sum e_i g(q_i)$$

*proof.* When $p$ is not a branch point, we will have $e_i = 1$ for all $i$ and $k = n$. In this case, the lemma simply restates (8.6.2). It follows by continuity for any point $p$ (see (8.4.1) **(ii)**. □

The next proposition is a secret that is essential for what follows.

smallpole **8.6.4. Proposition.** *With notation as above, let $g$ be a rational function on $Y$, and suppose that for $i = 1, ..., k$, $g$ has a pole of order at most $e_i - 1$ at the point $q_i$. Then the trace $\mathrm{tr}(g)$ is a regular function on $X$ at $p$.*

*proof.* Say that $p$ is the point $x = 0$ of $X$. Let $h = xg$. Since $x$ has a zero of order $e_i$ at $q_i$ and $g$ has a pole of order at most $e_i - 1$, $h$ is a regular function at $q_i$, and $h(q_i) = 0$, for $i = 1, ..., k$. The trace is a regular function at $p$, and it vanishes at $p$. Therefore $\mathrm{tr}(h)/x$ is also a regular function at $p$. On the other hand, since $\mathrm{tr}$ is $\mathcal{O}_X$-linear, $\mathrm{tr}(h) = x\,\mathrm{tr}(g)$. Therefore $\mathrm{tr}(g)$ is also regular at $p$. □

For example, when $Y$ is the locus $y^e = x$, multiplication by $\zeta = e^{2\pi i/e}$ permutes the sheets of $Y$ over $X$. The trace of $y^r$ is

sumzeta (8.6.5)
$$\sum_i \zeta^{ri} y^r$$

It is zero unless $r \equiv 0$ modulo $e$.

traced **(8.6.6) trace of a differential**

Let $Y \to X$ denote a map of a smooth projective curve $Y$ to $\mathbb{P}^1$ as before, and let $K$ and $F$ denote the function fields of $Y$ and $X$, respectively. The trace for differentials, which we are about to define, will be denoted by $\tau$.

We first define this trace for differentials of the function field $K$. Because the $\mathcal{O}_Y$-module $\Omega_Y$ is invertible, the module $\Omega_K$ of $K$-differentials is a free $K$-module of rank one (see (8.5.4) **(ii)**). Any nonzero differential will form a $K$-basis. So one can write an element $\alpha$ of $\Omega_K$ uniquely in the form

$$\alpha = g\,dx$$

where $x$ is the coordinate variable in $X$ and $g$ is an element of $K$. The trace $\tau$ is defined by

deftrdif (8.6.7)
$$\tau(g\,dx) = \mathrm{tr}(g)dx$$

where $\mathrm{tr}(g)$ is the trace of the function $g$. Thus $\tau$ is an $F$-linear map $\Omega_K \to \Omega_F$.

regdiff **8.6.8. Definition.** A differential $\alpha$ of $\Omega_K$ is *regular* at a point $q$ of $Y$ if there is an affine open neighborhood $Y' = \operatorname{Spec} B$ of $q$ such that $\alpha$ is an element of $\Omega_B$.

tracereg **8.6.9. Proposition.** *Let $p$ be a point of $X$ and let $q_1, ..., q_k$ be the points of $Y$ that lie over $p$. If a differential $\alpha$ on $Y$ is regular at the points $q_1, ..., q_k$, its trace $\tau(\alpha)$ is regular at $p$.*

*proof.* We write $\alpha = gdx$, where $g$ is a rational function on $Y$, an element of $K$. So $\tau(\alpha) = \mathrm{tr}(g)dx$, and we want to show that $\mathrm{tr}(g)$ is a regular function at $p$. According to Proposition 8.6.4, it suffices to show that $g$ has poles of orders at most $e_i - 1$ at the points $q_i$. We drop the subscript $i$. Let $q$ be a point that lies over $p$.

We may assume that $X$ and $Y$ are affine, $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$, and that the maximal ideals at $p$ and $q$ are principal ideals, say $\mathfrak{m}_p = xA$ and $\mathfrak{m}_q = yB$. Let $e = \mathrm{v}_q(x)$ be the ramification index at $q$, so that $x = uy^e$, where $\mathrm{v}_q(u) = 0$. We may localize to make $u$ a unit. Since $dy$ generates $\Omega_B$, $du = v\,dy$ for some regular function $v$. (Analytically, $v = \frac{du}{dy}$.) Then

dx (8.6.10)
$$dx = y^e du + y^{e-1}u\,dy = y^{e-1}\big(yv + u\big)dy$$

Since $u$ is a unit, so is $yv + u$. The formula shows that $dy = h\,dx$, where $h$ has a pole of order $e - 1$. Then if $\alpha = g\,dy$ is regular at $q$, then $hg$ will have a pole of order at most $e - 1$ at $q$. □

For example, if $x = y^e$, then $dx = ey^{e-1}dy$.

omegaYandX **8.6.11. Corollary.** *The trace map (8.6.7) defines a homomorphism of $\mathcal{O}_X$-modules $\Omega_Y \xrightarrow{\tau} \Omega_X$.* □

We define a map

defeps (8.6.12)
$$\Omega_Y \xrightarrow{\epsilon} \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{O}_Y, \Omega_X)$$

by sending a differential $\alpha$ to the map $\mathcal{O}_Y \xrightarrow{\epsilon_\alpha} \Omega_X$ defined by

$$\epsilon_\alpha(f) = \tau(f\alpha)$$

traceisom **8.6.13. Theorem.** *This map is an isomorphism of $\mathcal{O}_Y$-modules.*

I wish that I had a conceptual proof of this theorem. One impediment the lack of a conceptual definition of a differential. My father, Emil Artin, said: "One can't really understand differentials, but one learns to work with them". Anyway, we prove the theorem by computation.

*proof.* Let $\mathcal{H} = \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{O}_Y, \Omega_X)$. We verify that the map $\Omega_Y \xrightarrow{\epsilon} \mathcal{H}$ is $\mathcal{O}_Y$-linear. Let $\alpha$ be a regular differential on the inverse image $Y'$ of an open subset $X'$ of $X$, and let $h$ be a regular function on $Y'$. We must show that $h\epsilon_\alpha = \epsilon_{(h\alpha)}$. This is true:

$$[h\epsilon_\alpha](f) = \epsilon_\alpha(hf)\,\tau(hf\alpha) = \tau(fh\alpha) = \epsilon_{(h\alpha)}(f)$$

Thus $\epsilon$ is a homomorphism of $\mathcal{O}_Y$-modules. We also know that $\Omega_Y$ is an invertible $\mathcal{O}_Y$-module. We note that $\mathcal{H}$ is is torsion-free, and it has rank one as $\mathcal{O}_Y$-module because it has the same rank as $\mathcal{O}_Y$ as $\mathcal{O}_X$-module. Thus $\mathcal{H}$ is invertible. Since the map $\epsilon$ is nonzero, it is injective. Then Proposition 8.2.4 shows that $CH = \Omega_Y(D)$ for some effective divisor $D$. To show that $\epsilon$ is an idomorphism, we show that $D = 0$, and to do this, we show that, with $q_1, ..., q_k$ lying over $p$ as before, there is a differential $\alpha$ on $Y$ with a simple pole at one of the points, say $q_1$, that is regular at the points $q_2, ..., q_k$, and whose $\tau(\alpha)$ isn't a regular differential at $p$. Then $\alpha$ is not a section of $\Omega_Y(D)$, but it is a section of $\Omega_Y(q_1)$. So the coefficient of $q_1$ in $D$ must be zero.

The logarithmic differential $\alpha = dx/x$, viewed as a differential on $Y$, has a pole of order $e_1$ at $q_1$, and its trace is $ndx/x$, which is not regular at $p$. But because $\alpha$ isn't regular at the points $q_2, ..., q_k$, we write this differential as a sum. The power $\mathfrak{m}_1^N$ and the product of the powers $\mathfrak{m}_2^N \cdots \mathfrak{m}_k^N$ are comaximal. So we can write $1 = g_1 + g_2$ with $g_1 \in \mathfrak{m}_1^N$ and $g_2 \in \mathfrak{m}_2^N \cdots \mathfrak{m}_k^N$. The trace $\mathrm{tr}(g_2)$ is a regular function at $p$, whose value $e_1 g_2(q_1)$ at $p$ is nonzero. If $N$ is large, $g_2\alpha$ has a pole at $q_1$, is regular at $q_2, ..., q_k$, and its trace is $n\,\tau(g_2)dx/x$, which has a pole at $p$. □

For example, if $y^e = x$, then $dx/x = e\,dy/y$.

## 8.7 Riemann-Roch

rroch

Let $X$ be a smooth projective curve, and let $\mathcal{M}$ be a locally free $\mathcal{O}_X$-module. We denote the module $\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{M}, \Omega_X)$ by $\mathcal{M}^\sharp$. This module is the *Serre dual* of $\mathcal{M}$. Since $\Omega_X$ is invertible, the Serre dual $\mathcal{M}^\sharp$ will be locally free, and it will have the same rank as $\mathcal{M}$. Moreover, the canonical map $\mathcal{M} \to (\mathcal{M}^\sharp)^\sharp$ will be an isomorphism. This follows from the discussion of **8.1.6** because $\mathcal{M}$ and $\mathcal{L}$ are locally free. It also follows that the Serre dual of $\mathcal{O}_X$ is $\mathcal{O}_X^\sharp = \Omega_X$ and that the Serre dual of $\Omega_X$ is $\Omega_X^\sharp = \mathcal{O}_X$.

dualcohom **8.7.1. Riemann-Roch Theorem (version 2).** *Let $X$ be a smooth projective curve, let $\mathcal{M}$ be a locally free $\mathcal{O}_X$-module, and let $\mathcal{M}^\sharp$ be its Serre dual. Then* $\dim H^1(X, \mathcal{M}) = \dim H^0(X, \mathcal{M}^\sharp)$*, and* $\dim H^0(X, \mathcal{M}) = \dim H^1(X, \mathcal{M}^\sharp)$.

Since $\mathcal{M}$ and $(\mathcal{M}^\sharp)^\sharp$ are isomorphic, the two assertions of the theorem are equivalent.

A more precise version of this theorem states that the vector spaces $H^1(X, \mathcal{M})$ and $H^0(X, \mathcal{M}^\sharp)$ are dual spaces. We omit the proof. The fact that the dimensions of these spaces are equal is enough for nearly every application. And of course, any complex vector spaces $V$ and $W$ whose dimensions are equal can be made into dual spaces by the choice of a bilinear form $V \times W \to \mathbb{C}$. The precise duality is useful only when one needs to analyze the cohomology map that corresponds to a homomorphism $\mathcal{M} \to \mathcal{N}$.

Our plan is to prove the theorem 8.7.1 directly for the case of the projective line. The structure of locally free modules on $\mathbb{P}^1$ is very simple, so this will be easy. We will deduce it for an arbitrary smooth projective curve $Y$ by projection to $\mathbb{P}^1$.

We discuss the projection to $\mathbb{P}^1$ first. Let $Y$ be a smooth projective curve, let $Y \xrightarrow{\pi} X = \mathbb{P}^1$ be a finite morphism, and let $\mathcal{M}$ be a locally free $\mathcal{O}_Y$-module. The Serre dual of $\mathcal{M}$, as defined above, is

$$\mathcal{M}_1^\sharp = \underline{\mathrm{Hom}}_{\mathcal{O}_Y}(\mathcal{M}, \Omega_Y)$$

However, we can also view $\mathcal{M}$ as a locally free $\mathcal{O}_X$-module by taking its direct image We are denoting this direct image by $\mathcal{M}$ too. Then we can form the Serre dual on $X$:

$$\mathcal{M}_2^\sharp = \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{M}, \Omega_X)$$

abouthom **8.7.2. Lemma.** *When regarded as an $\mathcal{O}_X$-module, the Serre dual $\mathcal{M}_1^\sharp$ is canonically isomorphic to $\mathcal{M}_2^\sharp$.*

*proof.* We map $\underline{\mathrm{Hom}}_{\mathcal{O}_Y}(\mathcal{M}, \Omega_Y)$ to $\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{M}, \Omega_X)$ by composition with the trace map for differentials. If $\mathcal{M} \xrightarrow{\varphi} \Omega_Y$ is an $\mathcal{O}_Y$-homomorphism, its image $\tau \circ \varphi$ is an $\mathcal{O}_X$-homomorphism $\mathcal{M} \to \Omega_X$. This gives us a map $\mathcal{M}_1^\sharp \xrightarrow{\tau\circ} \mathcal{M}_2^\sharp$. To prove that the map is an isomorphism, it suffices to look on an affine open covering of $X$. Then since $\mathcal{M}$ is locally free, it suffices to prove that the map is an isomorphism when $\mathcal{M}$ is the structure sheaf $\mathcal{O}_Y$. In that case, the functorial map is the trace map $\underline{\mathrm{Hom}}_{\mathcal{O}_Y}(\mathcal{O}_Y, \Omega_Y) \xrightarrow{\tau} \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{O}_Y, \Omega_X)$. The fact that this map is an isomorphism is Theorem 8.6.13. □

We now drop the subscripts from $\mathcal{M}^\sharp$.

dualitymaps **8.7.3. Corollary.** *Let $Y \xrightarrow{\pi} X$ be a finite morphism of projective curves, and let $\mathcal{M}$ be a locally free $\mathcal{O}_Y$-module. Then $H^q(Y,\mathcal{M}) \approx H^q(X,\mathcal{M})$ and $H^q(Y,\mathcal{M}^\sharp) \approx H^q(X,\mathcal{M}^\sharp)$.*

See Lemma 7.4.17. □

This corollary shows that it suffices to prove Theorem 8.7.1 for the case that $Y = \mathbb{P}^1$.

dualityfor pone **(8.7.4) Riemann-Roch for the projective line**

Here $X$ denotes the projective line $\mathbb{P}^1$. We apply the theorem of Birkhoff and Grothendieck: Every locally free $\mathcal{O}_X$-module $\mathcal{M}$ is a direct sum of twisting sheaves. Thus it suffices to verify the Riemann-Roch Theorem for the twisting sheaves $\mathcal{O}_X(n)$.

omegapone **8.7.5. Lemma.** *The module $\Omega_X$ on $X$ is isomorphic to the twisting sheaf $\mathcal{O}_X(-2)$.*

*proof.* On the standard open set $\mathbb{U}^0 = \operatorname{Spec}\mathbb{C}[x]$, the module of differentials is generated by $dx$. Let $z = x^{-1}$ be the coordinate of $X$ on $\mathbb{U}^1$. Then $dx = d(z^{-1}) = -z^{-2}dz$ describes the differential $dx$ on $\mathbb{U}^1$. It has a pole of order 2 at the point $p_\infty$ at infinity. So $dx$ is a section of $\Omega_X(2p_\infty)$. This section is nowhere zero, and defines isomorphisms $\mathcal{O}_X \approx \Omega_X(2p_\infty)$ and $\Omega_X \approx \mathcal{O}_X(-2p_\infty) \approx \mathcal{O}_X(-2)$. □

The Serre dual of $\mathcal{O}_X(n)$ is then $\underline{\operatorname{Hom}}_{\mathcal{O}_X}(\mathcal{O}_X(n),\Omega_X) \approx \underline{\operatorname{Hom}}_{\mathcal{O}_X}(\mathcal{O}_X(n),\mathcal{O}_X(-2))$.

twisthom **8.7.6. Lemma.** *Let $Y$ be a projective variety, and let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{O}$-modules, Then $\underline{\operatorname{Hom}}_{\mathcal{O}}(\mathcal{M},\mathcal{N})$ is canonically isomorphic to $\underline{\operatorname{Hom}}_{\mathcal{O}}(\mathcal{M}(r),\mathcal{N}(r))$.*

*proof.* Say that $Y$ is embedded into the projective space $\mathbb{P}$. By definition, $\mathcal{M}(r) = \mathcal{M} \otimes_{\mathcal{O}_\mathbb{P}} \mathcal{O}_\mathbb{P}(n)$. A homomorphism $\mathcal{M} \xrightarrow{\varphi} \mathcal{N}$ defines a homomorphism $\mathcal{M}(r) \xrightarrow{\varphi\otimes 1} \mathcal{N}(r)$. This gives us a map $\underline{\operatorname{Hom}}_{\mathcal{O}}(\mathcal{M},\mathcal{N}) \to \underline{\operatorname{Hom}}_{\mathcal{O}}(\mathcal{M}(r),\mathcal{N}(r))$. Twisting with $\mathcal{O}(-r)$ inverts this map. □

It follows that

$$\mathcal{O}_X(n)^\sharp \approx \underline{\operatorname{Hom}}_{\mathcal{O}_X}(\mathcal{O}_X(n),\mathcal{O}_X(-2)) \approx \underline{\operatorname{Hom}}_{\mathcal{O}_X}(\mathcal{O}_X,\mathcal{O}_X(-2-n)) \approx \mathcal{O}_X(-2-n)$$

So to prove Riemann-Roch for $X = \mathbb{P}^1$, we must show that

$$\dim H^0(X,\mathcal{O}(n)) = \dim H^1(X,\mathcal{O}(-2-n)) \quad \text{and} \quad \dim H^1(X,\mathcal{O}(n)) = \dim H^0(X,\mathcal{O}(-2-n))$$

This follows from the computation of cohomology on projective space (see Theorem 7.5.4). □

## 8.8 Genus

genussm-curve There are three closely related numbers associated to a smooth projective curve $Y$. The first two are its topological *genus* $g$, and its *arithmetic genus* $p_a = \dim H^1(Y,\mathcal{O}_Y)$. The topological genus is defined, because $Y$ is connected. The third number is the degree $\delta$ of the module of differentials $\Omega_Y$, which is the difference $z-p$ of the numbers of zeros and of poles of a differential, or the degree of a canonical divisor (see 8.5.7).

genusgenus **8.8.1. Theorem.** *Let $Y$ be a smooth projective curve.*
**(i)** *The topological genus and the arithmetic genus of $Y$ are equal: $g = p_a$.*
**(ii)** *The degree of the module $\Omega_Y$ of differentials on $Y$ is $\delta = 2p_a - 2$, which is also equal to $2g-2$.*

Using a canonical divisor $K$, one can write the Riemann-Roch Theorem for $\mathcal{O}(D)$ in the form

RROD **8.8.2. Corollary.** $\dim H^0(Y,\mathcal{O}(D)) = \dim H^1(Y,\mathcal{O}(K-D))$ *and* $\dim H^0(Y,\mathcal{O}(K-D)) = \dim H^1(Y,\mathcal{O}(D))$. □

*proof of the theorem.* **(ii)** Riemann-Roch tells us that $\dim H^0(Y,\Omega_Y) = \dim H^1(Y,\mathcal{O}_Y) = p_a$ and that $\dim H^1(Y,\Omega_Y) = \dim H^0(Y,\mathcal{O}_Y) = 1$ (see (8.7.1). So $\chi(\Omega_Y) = p_a - 1$. Since we also know that $\chi(\Omega_Y) = \delta + 1 - p_a$ (8.3.6), $\delta = 2p_a - 2$, as expected.

**(i)** We choose a map $Y \xrightarrow{\pi} X$ to the projective line. The topological Euler characteristic of $Y$ is $e(Y) = 2 - 2g$. As we have noted before, $e(Y)$ can be computed in terms of the branching data for the covering $Y$ over $X$:

$$2 - 2g = e(Y) = ne(X) - \sum(e_i - 1) = 2n - \sum(e_i - 1)$$

Next, we determine the degree of $\Omega_Y$ by computng the divisor of the differential $dx$ on $Y$. Let $q$ be a point of $Y$ with ramification index e, possibly $e = 1$. We choose coordinates so that its image $p$ in $X$ is the point $x = 0$. Let's suppose that the point at infinity isn't a branch point. Then if $y$ is a local generator for the maximal ideal $\mathfrak{m}_q$, we will have $x = uy^e$, and $dx$ has a zero of order $e - 1$ at $q$ (8.6.10).

On $X$, $dx$ has a pole of order 2 at $\infty$. Then if $n$ is the degree of $Y$ over $X$, there will be $n$ points on $Y$ at which $dx$ has a pole of order 2. The degree of $\Omega_Y$ is therefore

$$\delta = \#\{zeros\} - \#\{poles\} = \sum(e_i - 1) - 2n = -e(Y) = 2g - 2$$

Thus $\delta 2g - 2$. Since we also know that $\delta = 2p_a - 2$, as claimed. $p_a = g$. □

The next corollary follows by duality from Corollary 8.3.5 **(v)** and **(vi)**.

Honezero **8.8.3. Corollary.** *Let $D$ be a divisor on a smooth projective curve $Y$ of genus $g$.*
**(i)** *If* $\deg\ D > 2g - 2$ *then* $H^1(Y, \mathcal{O}(D)) = 0$.
**(ii)** *If* $\deg\ D \le g - 2$, *then* $H^1(Y, \mathcal{O}(D)) \neq 0$. □

## 8.9 Curves of Genus One

lowgenus We discuss curves of genus zero or one here, leaving thwe analysis of higher genus curves as exercises.

The next proposition will be convenient for use in this section. It is an example of the general principle that a finite sequence of computations done in a localization can be made in a simple localization.

pointsinfibre **8.9.1. Proposition.** *Let $Y \xrightarrow{u} X$ be a morphism of varieties whose image contains a nonempty open subset of $X$, and let $K$ and $L$ be the function fields of $X$ and $Y$, respectively. Suppose that $L$ is a finite extension of $K$ of degree $n$. Then there are open subsets $X' \subset X$ and $Y' = u^{-1}X' \subset Y$ such that all fibres of $Y'$ over $X'$ have order $n$.*

*proof.* We may replace $X$ and $Y$ by arbitrary nonempty open subsets, and we may do this finitely often. Thus we may assume that $X$ and $Y$ are affine, say $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$. The fraction fields of $A$ and $B$ will be $K$ and $L$, respectively. Let $B \otimes_A K = BS^{-1}$, where $S$ is the multiplicative system of nonzero elements of $A$. Since $L$ is a finite extension of $K$ and $B \otimes_A K \subset L$, $B \otimes_A K$ is a field. It is therefore equal to $L$. This means that every element of $L$ can be written as a fraction $b/s$, with $b \in B$ and $s \in A$.

The extension $L$ of $K$ can be generated by one element $\beta$. Let $f(y)$ be the monic irreducible polynomial for $\beta$ over $K$, so that $L = K[\beta] = K[y]/(f)$. Localizing, we may assume that the coefficients of $f$ are in $A$, and that $\beta$ is in $B$. Let $B_0 = A[\beta] = A[y]/(f)$. Then $B_0 \subset B$.

Let $b_1, ..., b_k$ be generators for the finite-type algebra $B$. They are in $L = K[\beta]$, and therefore in a simple localization of $A[\beta] = B_0$ by an element $s$ of $A$. Localizing again, we may assume that they are elements of $B_0$. Then $B_0 = B$. We keep the notation $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ for these localized varieties.

Say that $f(y) = y^n + a_{n-1}y^{n-1} + \cdots + a_0$, and let $p$ be a point of $X$. The fibre of $Y$ over the point $p$ is the set of roots of the polynomial $\overline{f}(y) = y^n + \overline{a}_{n-1}y^{n-1} + \cdots + \overline{a}_0$, where $\overline{a}_i = a_i(p)$. There will be $n$ points in the fibre, provided that the discriminant of $\overline{f}$ isn't zero. Since $f$ is a polynomial with coefficients in the field $K$, its discriminant isn't identically zero (Proposition **??**). When we localize by inverting this discriminant, all fibres will have order $n$. □

genuszero **(8.9.2)** **curves of genus zero**

Let $Y$ be a smooth projective curve of genus $g = 0$. So $H^1(Y, \mathcal{O}_Y) = 0$. Let $p$ be a point of $Y$. The exact sequence

$$0 \to \mathcal{O}_Y \to \mathcal{O}_Y(p) \to \epsilon \to 0$$

gives us an exact cohomology sequence

$$0 \to H^0(Y, \mathcal{O}_Y) \to H^0(Y, \mathcal{O}_Y(p)) \to H^0(Y, \epsilon) \to 0$$

The zero at the end is due to the fact that $H^1(Y, \mathcal{O}_Y) = 0$. Therefore $\dim H^0(Y, \mathcal{O}_Y(p)) = 2$. We choose a basis $(x, 1)$ for $H^0(Y, \mathcal{O}_Y(p))$, 1 being the constant function. This basis defines a point with values in the function field $K$, and therefore a morphism $Y \xrightarrow{\varphi} \mathbb{P}^1$. Because $x$ has just one pole of order 1, it takes every value exactly once. Therefore $\varphi$ is bijective. It is a map of degree 1. The function fields $F$ of $\mathbb{P}^1$ and $K$ of $Y$ are isomorphic. The embedding of $Y$ into a projective space $\mathbb{P}^r$ that we assume given defines a point of $\mathbb{P}^r$ with values in $K$, which gives us a point with values in the isomorphic field $F$. The point with values in $F$ defines a morphism $\mathbb{P}^1 \to \mathbb{P}^r$ that inverts $\varphi$. So $\varphi$ is an isomorphism.

gzero **8.9.3. Corollary.** *Every curve of genus zero is isomorphic to the projective line* $\mathbb{P}^1$. □

genusone **(8.9.4) curves of genus one**

Let $Y$ be a smooth projective curve of genus 1. Riemann-Roch tells us that

$$\chi(\mathcal{O}(D)) = \deg\, D$$

The degree of a canonical divisor $K$ is $2g - 2 = 0$. Since $\dim H^0(Y, \mathcal{O}(K)) = \dim H^1(Y, \mathcal{O}) = 1$, $\mathcal{O}(K)$ has a nonzero global section. This section cannot vanish anywhere, so it defines an isomorphism $\mathcal{O} \to \mathcal{O}(K)$. The module of differentials is a free module of rank one.

Let $p$ be a point of $Y$. Then $K - rp$ has negative degree. Therefore

$$\dim H^0(Y, \mathcal{O}(K - rp)) = \dim H^1(Y, \mathcal{O}(rp)) = 0$$

- If $r \geq 1$, then $\dim H^0(Y, \mathcal{O}_Y(rp)) = r$ and $H^1(Y, \mathcal{O}_Y(rp)) = 0$.

(This conclusion could be derived from version 1 of the Riemann-Roch Theorem.)

The basis $(1)$ for $H^0(Y, \mathcal{O}_Y)$ is also a basis for $H^0(Y, \mathcal{O}_Y(p))$. We choose a basis $(x, 1)$ for $H^1(Y, \mathcal{O}_Y(2p))$, which has dimension 2. Then because it is not a section of $\mathcal{O}(p)$, $x$ has a pole of order precisely 2 at $p$. Next, we choose a basis $(x, y, 1)$ for $H^1(Y, \mathcal{O}_Y(3p))$. So $x$ and $y$ are functions with poles of orders 2 and 3, respectively, at $p$, and no other poles. Having done this, $(x, y, 1)$ determines a point of $\mathbb{P}^2$ with values in $K$, and a morphism $Y \to \mathbb{P}^2$ that sends a point $q$ distinct from $p$ to $(x(q), y(q), 1)$.

To determine the image of $p$, we multiply by $y^{-1}$ to normalize the second coordinate to 1, obtaining the equivalent vector $(xy^{-1}, 1, y^{-1})$. The rational function $xy^{-1}$ has a simple zero at $p$ and $y^{-1}$ has a zero of order 3. Evaluating at $p$, we see that the image of $p$ is the point $(0, 1, 0)$.

Let $\ell$ be a line $\{ax + by + cz = 0\}$ in $\mathbb{P}^2$, with $b \neq 0$. On $Y$, $ax + by + c$ is a function with a pole of order 3 at $p$ and no other pole. It takes the value 0 three times, counted with multiplicity. This means that $\ell$ meets the image of $Y$ in three points. So the image of $\varphi$ is a cubic curve in the plane. There must be a cubic relation among the functions $x, y, 1$.

To find this cubic relation, we determine bases for the space $H^0(Y, \mathcal{O}_Y(rp))$, when $r = 4, 5, 6$. The dimension of that space is $r$. Setting $r = 4$, we note that $x^2$ has a pole of order 4 at $p$. It is not a combination of $x, y, 1$, which have poles orders $\leq 3$. Therefore $(1, x, y, x^2)$ is a basis for $H^0(Y, \mathcal{O}_Y(4p))$. Continuing, $(1, x, y, x^2, xy)$ is a basis for $H^0(Y, \mathcal{O}_Y(5p))$. But there are two monomials in $x, y$ with pole of orders 6, namely $x^3$ and $y^2$. Since $\dim H^0(Y, \mathcal{O}_Y(6p)) = 6$, there is a linear relation among the seven monomials $1, x, y, x^2, xy, x^3, y^2$. This gives us a cubic equation satisfied by the image $Y'$.

The cubic relation has the form

$$cy^2 + (a_1x + a_3)y + (a_0x^3 + a_2x^2 + a_4x + a_6) = 0.$$

The coefficients of $y^2$ and $x^3$ aren't zero, so we may assume that $c = 1$. We "complete the square", adding a combination of 1 and $x$ to $y$ to eliminate the linear term in $y$ from this relation. Then we replace $x$ by a combination of $1, x$ to eliminate the quadratic term in $x$ and to normalize $a_0$ to 1. Bringing the terms in $x$ to the other side of the equation, we are left with a cubic relation

$$y^2 = x^3 + a_4x + a_6.$$

The coefficients $a_4$ and $a_6$ will have changed, of course.

The image of $Y$ is the cubic plane curve $Y'$ defined by the homogenized equation $y^2z = x^3+a_4xz^2+a_6z^3$.

As it happens, the genus of a cubic curve in $\mathbb{P}^2$ is equal to 1 (see Section **??**). Using this fact, one can show that $Y$ maps isomorphically to $Y'$.

Summing up, every curve of genus 1 is isomorphic to a cubic curve in $\mathbb{P}^2$.

genusone-froup

### (8.9.5) the group law on a curve of genus 1

A smooth curve $Y$ of genus 1, an *elliptic curve* has a group law that is uniquely determined, once one chooses a point to be the identity element.

We choose a point of $Y$ and label it $o$. We'll write the group law that is to be defined additively, using the symbol $p \oplus q$ for the sum. We do this to distinguish the sum $p \oplus q$, which is a point of $Y$, from the divisor $p + q$.

Let $p$ and $q$ be points of $Y$. To define $p \oplus q$, we compute the cohomology of $\mathcal{O}_Y(p+q-o)$. Riemann-Roch shows that $\dim H^0(Y, \mathcal{O}_Y(p+q-o)) = 1$ and that $H^1(Y, \mathcal{O}_Y(p+q-o)) = 0$. There is a nonzero function $f$, unique up to scalar factor, with simple poles at $p$ and $q$ and a zero at $o$. This function has exactly one other zero. That zero is defined to be the sum $s = p \oplus q$ in the group.

In terms of linearly equivalent divisors, $s$ is the unique point such that the divisor $s$ is linearly equivalent to $p + q - o$, or such that $p + q$ and $o + s$ are linearly equivalent. Note that this is a commutative law of composition on $Y$. Let'sdenote linear equivalence by the symbol $\sim$. So $D \sim E$ mreans that $D - E$ is the divisor of a function. To verify the associative law, we let $p, q, r$ be points of $Y$. Then $(p \oplus q) \oplus r$ is obtained this way: We let $s = p \oplus q$ , so that $s \sim p + q - o$. Then we let $t = s \oplus r$, so that $t \sim s + r - o$. Combining, $t \sim p + q - o + r - o = p + q + r - 2o$. This determines the point $t$, and computation of $p \oplus (q \oplus s)$ leads to the same answer.

Finally, we check that a point $p$ has an inverse by solving the equation $p \oplus q = o$ for $q$. We need $q$ so that $o \sim p + q - o$, or that $q \sim 2o - p$. As before, one finds that $\dim H^0(Y, \mathcal{O}_Y(2o - p)) = 1$, so there is a unique solution for $q$.