Chapter 7 COHOMOLOGY

Determining the rational functions on a projective curve X that have poles at some points is classical problem of algebraic geometry. One can solve this problem easily when X is the projective line. Let t be an affine coordinate. We may ask for rational functions that have poles of order at most 1 at the point $t = 0$, of order at most 2 at the point $t = 1$, and no other poles. One such function is $f = 1/t(t-1)^2$. This function has a zero of order 3 at infinity, so its product with a polynomial of degree ≤ 3 has no pole at infinity. The space of functions with the given poles has the basis (f, tf, t^2f, t^3f). However, the problem is much harder when X is a curve of large degree. It is usually difficult to determine those functions explicitly, and one is happy when one can determine the dimension of the space they span. The main tool for this, cohomology, is the topic of this chapter.

7.1 Complexes

complexes

We will need to work with complexes. A *complex* of vector spaces $V^\bullet$ is a sequence of homomorphisms

complexone (7.1.1)
$$\cdots \to V^{n-1} \xrightarrow{d^{n-1}} V^n \xrightarrow{d^n} V^{n+1} \xrightarrow{d^{n+1}} \cdots$$

such that the composition $d^n d^{n-1}$ of adjacent maps is zero, i.e., that the image of d^{n-1} is contained in the kernel of d^n. The q-dimensional *cohomology* $\mathbf{h}^q$ of a complex $V^\bullet$ is

cohcoplxone (7.1.2)
$$\mathbf{h}^q(V^\bullet) = (\text{ker } d^q)/(\text{im } d^{q-1}).$$

An exact sequence is a complex whose cohomology is zero.

Any homomorphism of vector spaces can be made into a complex by adding zeros:

$$\cdots 0 \to V^0 \xrightarrow{d^0} V^1 \to 0 \cdots$$

For such a complex, $\mathbf{h}^0 = \text{ker } d^0$, $\mathbf{h}^1 = \text{coker } d^0$, and $\mathbf{h}^q = 0$ for $q \neq 0, 1$.

We will also need the Snake Lemma, which we state here without proof.

snake **7.1.3. Proposition.** *Suppose given a diagram*

$$\begin{array}{ccccccc}
 & & V & \xrightarrow{u} & V' & \longrightarrow & V'' & \longrightarrow & 0 \\
 & & {\scriptstyle f}\downarrow & & {\scriptstyle f'}\downarrow & & {\scriptstyle f''}\downarrow & & \\
0 & \longrightarrow & W & \longrightarrow & W' & \xrightarrow[v]{} & W'' & &
\end{array}$$

whose rows are exact sequences. Let K, K', K'' and C, C', C'' denote the kernels and cokernels of f, f', and f'', respectively.
(i) (kernel is left exact) *The kernels form an exact sequence*

$$K \to K' \to K''.$$

If u is injective, the sequence

$$0 \to K \to K' \to K''$$

is exact.
(ii) (cokernel is right exact) *The cokernels form an exact sequence*

$$C \to C' \to C''.$$

If v is surjective, the sequence

$$C \to C' \to C'' \to 0$$

is exact.
(iii) (Snake Lemma) *There is a canonical homomorphism $K'' \xrightarrow{d} C$ that combines with the above sequences to form an exact sequence*

$$K \to K' \to K'' \xrightarrow{d} C \to C' \to C''.$$

If u is injective and/or v is surjective, the sequence remains exact with some zeros at the ends. □

A *map* $V^\bullet \xrightarrow{\varphi} V'^\bullet$ of complexes is a collection of homomorphisms $V^n \xrightarrow{\varphi^n} V'^n$ making a diagram

$$\begin{array}{ccccccccc}
\longrightarrow & V^{n-1} & \xrightarrow{d^{n-1}} & V^n & \xrightarrow{d^n} & V^{n+1} & \longrightarrow & \cdots \\
 & {\scriptstyle\varphi^{n-1}}\downarrow & & {\scriptstyle\varphi^{n}}\downarrow & & {\scriptstyle\varphi^{n+1}}\downarrow & & \\
\longrightarrow & V'^{n-1} & \xrightarrow{d'^{n-1}} & V'^n & \xrightarrow{d'^n} & V'^{n+1} & \longrightarrow & \cdots
\end{array}$$

A map of complexes induces maps on cohomology

$$\mathbf{h}^q(V^\bullet) \to \mathbf{h}^q(V'^\bullet)$$

because φ maps $\ker\, d^q$ to $\ker\, d'^q$ and $\text{im}\, d^q$ to $\text{im}\, d'^q$.
A sequence of maps of complexes

exseqcplx (7.1.4)
$$\cdots \to V^\bullet \xrightarrow{\varphi} V'^\bullet \xrightarrow{\psi} V''^\bullet \to \cdots$$

is *exact* if the sequence

$$\cdots \to V^q \xrightarrow{\varphi^q} V'^q \xrightarrow{\psi^q} V''^q \to \cdots$$

is exact for every q.

cohcplx **7.1.5. Proposition.** *Let $0 \to V^\bullet \to V'^\bullet \to V''^\bullet \to 0$ be a short exact sequence of complexes. For every q, there are maps $\mathbf{h}^q(V''^\bullet) \xrightarrow{\delta^q} \mathbf{h}^{q+1}(V^\bullet)$ such that the sequence*

$$\cdots \to \mathbf{h}^q(V^\bullet) \to \mathbf{h}^q(V'^\bullet) \to \mathbf{h}^q(V''^\bullet) \xrightarrow{\delta^q} \mathbf{h}^{q+1}(V^\bullet) \to \cdots$$

is exact.

This long exact sequence is called the *cohomology sequence* associated to the short exact sequence of complexes.

snakecohomology **7.1.6. Example.** The Snake Lemma provides an example of the cohomology sequence. Suppose given a diagram as in (7.1.3), and that u is injective and v is surjective. We make the map f of this diagram into a complex

$$0 \to V \xrightarrow{f} W \to 0,$$

and we do the analogous thing for the maps f' and f''. Then the diagram becomes a short exact sequence of complexes whose cohomology sequence is the one given by the Snake Lemma. □

proof of Proposition 7.1.5. Suppose given a complex $V^\bullet$. Let K^q and I^q denote the kernel and image of the map d^q, and let C^q denote the cokernel of the map d^{q-1}: $C^q = V^q/I^{q-1}$. So $\mathbf{h}^q(V^\bullet) = K^q/I^{q-1}$ is a quotient of K^q and a subspace of C^q.

The map d^q can be written as a composition of three maps:

$$V^q \xrightarrow{\pi^q} C^q \xrightarrow{f^q} K^{q+1} \xrightarrow{i^{q+1}} V^{q+1}$$

Here π^q is the canonical projection, i^{q+1} is the inclusion, and the map f^q is induced from d^q. The maps f^q provide two ways to identify the cohomology of the complex:

hiskerand-coker (7.1.7)
$$\mathbf{h}^q(V^\bullet) = \ker f^q \quad \text{and} \quad \mathbf{h}^{q+1}(V^\bullet) = \operatorname{coker} f^q.$$

Suppose given a short exact sequence of complexes $0 \to V^\bullet \to V'^\bullet \to V''^\bullet \to 0$ as in the proposition. When we apply (7.1.7) and the Snake Lemma to the diagram

$$\begin{array}{ccccccc}
 & & C^q & \longrightarrow & C'^q & \longrightarrow & C''^q & \longrightarrow & 0 \\
 & & {\scriptstyle f^q}\downarrow & & {\scriptstyle f'^q}\downarrow & & {\scriptstyle f''^q}\downarrow & & \\
0 & \longrightarrow & K^{q+1} & \longrightarrow & K'^{q+1} & \longrightarrow & K''^{q+1} & &
\end{array}$$

we obtain an exact sequence

$$\mathbf{h}^q(V^\bullet) \to \mathbf{h}^q(V'^\bullet) \to \mathbf{h}^q(V''^\bullet) \xrightarrow{\delta^q} \mathbf{h}^{q+1}(V^\bullet) \to \mathbf{h}^{q+1}(V'^\bullet) \to \mathbf{h}^{q+1}(V''^\bullet)$$

The cohomology sequence is obtained by splicing these sequences together. □

The coboundary maps δ^q in cohomology sequences are related in a natural way. If

$$\begin{array}{ccccccccc}
0 & \longrightarrow & V^\bullet & \longrightarrow & V'^\bullet & \longrightarrow & V''^\bullet & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & W^\bullet & \longrightarrow & W'^\bullet & \longrightarrow & W''^\bullet & \longrightarrow & 0
\end{array}$$

is a diagram of maps of complexes whose rows are short exact sequences, then the diagrams below commute:

$$\begin{array}{ccc}
\mathbf{h}^q(V''^\bullet) & \xrightarrow{\delta^q} & \mathbf{h}^{q+1}(V^\bullet) \\
\downarrow & & \downarrow \\
\mathbf{h}^q(W''^\bullet) & \xrightarrow{\delta^q} & \mathbf{h}^{q+1}(W^\bullet)
\end{array}$$

So there is a map of cohomology sequences associated to the map of short exact sequences.

With these long cohomology sequences, the set of functors $\{\mathbf{h}^q\}$ becomes what is called a *cohomological functor* on the category of complexes.

One more definition: Let $V^\bullet : \ 0 \to V^0 \to V^1 \to \cdots \to V^n \to 0$ be a complex of finite dimensional vector spaces with finitely many nonzero terms. Its *Euler characteristic* $\chi(V^\bullet)$ is the alternating sum $\sum(-1)^q \dim V^q$.

The proof of the next Proposition makes a good exercise.

altsum **7.1.8. Proposition.** *With notation as above,*

$$\chi(V^\bullet) = \sum(-1)^q \dim V^q = \sum(-1)^q \dim \mathbf{h}^q(V^\bullet).$$

□

cohqcoh

7.2 Cohomology of $\mathcal{O}$-Modules

Cohomology can be defined for any coefficient sheaf, but here we will define it only for $\mathcal{O}$-modules. This simplifies the construction. Anyhow, the Zariski topology is useless for cohomology with most other coefficients.

Let X be a variety, let $\mathcal{O} = \mathcal{O}_X$, and let $\mathcal{M}$ be a $\mathcal{O}$-module. The *zero-dimensional cohomology* of $\mathcal{M}$ is the space $\mathcal{M}(X)$ of global sections. When speaking of cohomology, one denotes this space by $H^0(X, \mathcal{M})$. The functor

$$(\mathcal{O}\text{-modules}) \to (\text{vector spaces})$$

that carries an $\mathcal{O}$-module $\mathcal{M}$ to $H^0(X, \mathcal{M})$ is left exact (Theorem 6.4.4). So if

SS (7.2.1)
$$0 \to \mathcal{M} \to \mathcal{N} \to \mathcal{P} \to 0$$

is an exact sequence of $\mathcal{O}$-modules, the associated sequence of global sections

globalsections (7.2.2)
$$0 \to H^0(X, \mathcal{M}) \to H^0(X, \mathcal{N}) \to H^0(X, \mathcal{P})$$

is exact. But as we have seen, the map $H^0(X, \mathcal{N}) \to H^0(X, \mathcal{P})$ may fail to be surjective when X isn't affine. The *cohomology* of $\mathcal{M}$ is a sequence

$$H^0(X, \mathcal{M}),\ H^1(X, \mathcal{M}),\ H^2(X, \mathcal{M}),\ ...$$

of vector spaces that compensates for the lack of exactness in the following way: To every short exact sequence $0 \to \mathcal{M} \to \mathcal{N} \to \mathcal{P} \to 0$ of $\mathcal{O}$-modules, there is associated a long exact *cohomology sequence*

HSS (7.2.3)
$$\begin{array}{c}
0 \to H^0(X, \mathcal{M}) \to H^0(X, \mathcal{N}) \to H^0(X, \mathcal{P}) \xrightarrow{\delta^0} \\
\xrightarrow{\delta^0} H^1(X, \mathcal{M}) \to H^1(X, \mathcal{N}) \to H^1(X, \mathcal{P}) \xrightarrow{\delta^1} \\
\cdots \quad \cdots \\
\xrightarrow{\delta^{q-1}} H^q(X, \mathcal{M}) \to H^q(X, \mathcal{N}) \to H^q(X, \mathcal{P}) \xrightarrow{\delta^q} \cdots
\end{array}$$

The coboundary maps δ^q in these sequences are analogous to the coboundary maps in the cohomology sequences of complexes. In fact, we will define the cohomology of an $\mathcal{O}$-module as the cohomology of a certain complex.

Given a diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{M} & \longrightarrow & \mathcal{N} & \longrightarrow & \mathcal{P} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{M}' & \longrightarrow & \mathcal{N}' & \longrightarrow & \mathcal{P}' & \longrightarrow & 0
\end{array}$$

whose rows are short exact sequences of $\mathcal{O}$-modules, the associated diagrams

deltadiagram (7.2.4)
$$\begin{array}{ccc}
H^q(X, \mathcal{P}) & \xrightarrow{\delta^q} & H^q(X, \mathcal{M}) \\
\downarrow & & \downarrow \\
H^q(X, \mathcal{P}') & \xrightarrow{\delta^{q-1}} & H^q(X, \mathcal{M}')
\end{array}$$

commute. So the map of short exact sequences induces a map of cohomology seuences.

This property makes the set of functors $H^0, H^1, ...$ into a *cohomological functor* from $\mathcal{O}$-modules to vector spaces. Though the property (7.2.4) is essential, it will appear explicitly only once, at the end of Section 7.4.

charprop

7.3 Characteristic Properties

The cohomology $H^q(X, \cdot)$ of $\mathcal{O}$-modules is characterized by three properties, of which the first two have already been mentioned:

charpropone (7.3.1)

- $H^0(X, \mathcal{M})$ is the space $\mathcal{M}(X)$ of global sections of $\mathcal{M}$.
- The sequence $H^0, H^1, H^2, \cdots$ is a cohomological functor on $\mathcal{O}$-modules.
- Let $Y \xrightarrow{f} X$ be the inclusion of an affine open subset Y into X, let $\mathcal{N}$ be an $\mathcal{O}_Y$-module, and let $f_*\mathcal{N}$ be its direct image on X. Then $H^q(X, f_*\mathcal{N})$ is zero for all $q > 0$.

If $\mathcal{M}$ is an $\mathcal{O}_X$-module. The third property, applied to the $\mathcal{O}_Y$-module $f^*\mathcal{M}$, asserts that $H^q(X, f_*f^*\mathcal{M}) = 0$ for all $q > 0$. The sheaf $f_*f^*\mathcal{M}$ is obtained from $\mathcal{M}$ by allowing poles on the complement of Y. So the third property asserts that, when one allows functions to have poles on the complement of an affine open set, the cohomology in positive dimensions becomes zero.

cohzeroaffine **7.3.2. Corollary.** *If X is an affine variety, $H^q(X, \mathcal{M}) = 0$ for all $\mathcal{O}$-modules $\mathcal{M}$ and all $q > 0$.*

This follows by applying the third characteristic property to the identity map. □

existcohom **7.3.3. Theorem.** *There exists a cohomology theory with the properties (7.3.1), and it is unique up to unique isomorphism.*

The proof of Theorem 7.3.3 will be given in the next section.

constrcoh

7.4 Construction

The proof of existence and uniqueness of cohomology is based on two facts:

- The intersection of affine open subsets of a variety is an affine open set (3.5.11), and
- A sequence $\cdots \to \mathcal{M} \to \mathcal{N} \to \mathcal{P} \to \cdots$ on a variety is exact if and only if, for every *affine* open subset U of X, the sequence of sections $\cdots \to \mathcal{M}(U) \to \mathcal{N}(U) \to \mathcal{P}(U) \to \cdots$ is exact. (This is the definition of exactness.)

We will work with a fixed, but arbitrary, affine open covering $\mathbb{U} = \{U^\nu\}$ of our variety X, and we let $\mathbb{U} \xrightarrow{j} X$ denote the family of inclusions $U^\nu \xrightarrow{j^\nu} X$. When we have shown that the cohomology is unique, we will know that it doesn't depend on our choice.

Let $\mathcal{M}$ be an $\mathcal{O}$-module, and let $\mathcal{R}^0$ or $\mathcal{R}^0_{\mathcal{M}}$ denote the $\mathcal{O}$-module $j_*j^*\mathcal{M} = \prod j^\nu_* j^{\nu *}\mathcal{M}$, where j^ν_* and $j^{\nu *}$ are the direct and inverse image functors that were defined in Chapter 6.

defcalr **7.4.1. Lemma.** **(i)** *If V is an open subset of X, then $\mathcal{R}^0_{\mathcal{M}}(V) = \prod \mathcal{M}(V \cap U^\nu)$. In particular, the space $\mathcal{R}^0_{\mathcal{M}}(X)$ of global sections of $\mathcal{R}^0_{\mathcal{M}}$ is the product $\prod \mathcal{M}(U^\nu)$.*

(ii) *The canonical map $\mathcal{M} \to \mathcal{R}^0_{\mathcal{M}} = j_*j^*\mathcal{M}$ of Proposition 6.5.10 is injective. Thus if $\mathcal{M}^1$ denotes the cokernel of that map, there is a short exact sequence of $\mathcal{O}$-modules $0 \to \mathcal{M} \to \mathcal{R}^0_{\mathcal{M}} \to \mathcal{M}^1 \to 0$.*

(iii) *For any cohomology theory with the characteristic properties, and for any $q > 0$, $H^q(X, \mathcal{R}^0_{\mathcal{M}}) = 0$*

proof. **(i)** This is seen by going through the definitions:

$$\mathcal{R}^0(V) = [j_*j^*\mathcal{M}](V) \;=\; \prod [j^\nu_* j^{\nu *}\mathcal{M}](V) = \prod [j^{\nu *}\mathcal{M}](V \cap U^\nu) = \prod \mathcal{M}(V \cap U^\nu).$$

(ii) On sections over an open set V, the map $\mathcal{M} \to \mathcal{R}^0_{\mathcal{M}}$ is the product of the restriction maps $\mathcal{M}(V) \to \mathcal{M}(V \cap U^\nu)$. Because the open sets U^ν cover X, the intersections $V \cap U^\nu$ cover V. The sheaf property of $\mathcal{M}$ tells us that the map $\mathcal{M}(V) \to \prod \mathcal{M}(V \cap U^\nu)$ is injective.

(iii) This follows from the third characteristic property. □

Rcminjective **7.4.2. Lemma.** **(i)** *A short exact sequence* $0 \to \mathcal{M} \to \mathcal{N} \to \mathcal{P} \to 0$ *of* $\mathcal{O}$*-modules embeds into a diagram*

ttdiagr (7.4.3)
$$\begin{array}{ccccc} \mathcal{M} & \longrightarrow & \mathcal{N} & \longrightarrow & \mathcal{P} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{R}^0_{\mathcal{M}} & \longrightarrow & \mathcal{R}^0_{\mathcal{N}} & \longrightarrow & \mathcal{R}^0_{\mathcal{P}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{M}^1 & \longrightarrow & \mathcal{N}^1 & \longrightarrow & \mathcal{P}^1 \end{array}$$

whose rows and columns are short exact sequences.
(ii) *The sequence of global sections* $0 \to \mathcal{R}^0_{\mathcal{M}}(X) \to \mathcal{R}^0_{\mathcal{N}}(X) \to \mathcal{R}^0_{\mathcal{P}}(X) \to 0$ *is exact.*

proof. **(i)** We are given that the top row of the diagram is a short exact sequence, and we have seen that the columns are short exact sequences. To show that the middle row

Rsequence (7.4.4)
$$0 \to \mathcal{R}^0_{\mathcal{M}} \to \mathcal{R}^0_{\mathcal{N}} \to \mathcal{R}^0_{\mathcal{P}} \to 0$$

is exact, we must show that if W is an affine open subset, the sections on W form a short exact sequence. The sections of $\mathcal{R}$ are explained in (7.4.1). Since products of exact sequences are exact, what is to be shown is that the sequence
$$0 \to \mathcal{M}(W \cap U^\nu) \to \mathcal{N}(W \cap U^\nu) \to \mathcal{P}(W \cap U^\nu) \to 0$$
is exact. This is true because $W \cap U^\nu$ is an intersection of affine opens, and is therefore affine. The fact that the bottom row of the diagram is a short exact sequence follows from the Snake Lemma.

(ii) The sequence of of global sections is the product of the sequences
$$0 \to \mathcal{M}(U^\nu) \to \mathcal{N}(U^\nu) \to \mathcal{P}(U^\nu) \to 0$$
These sequences are exact because the open sets U^ν are affine. □

Uniqueness of cohomology: Suppose given a cohomology theory with the characteristic properties (7.3.1), and let $\mathcal{M}$ be an $\mathcal{O}$-module. Then $H^q(X, \mathcal{R}^0_{\mathcal{M}}) = 0$ if $q > 0$ (Lemma 7.4.1 **(iii)**). Therefore the cohomology sequence associated to the sequence $0 \to \mathcal{M} \to \mathcal{R} \to \mathcal{M}_1 \to 0$ gives us an exact sequence

hone (7.4.5)
$$0 \to H^0(X, \mathcal{M}) \to H^0(X, \mathcal{R}^0_{\mathcal{M}}) \to H^0(X, \mathcal{M}^1) \xrightarrow{\delta^0} H^1(X, \mathcal{M}) \to 0,$$

and it gives us isomorphisms

hq (7.4.6)
$$H^q(X, \mathcal{M}^1) \xrightarrow{\delta^q} H^{q+1}(X, \mathcal{M})$$

for every $q > 0$. The first three terms of the sequence (7.4.5), and the arrows connecting them, depend on our choice of covering of X, but the important point is that they don't depend on the cohomology theory. So the sequence determines $H^1(X, \mathcal{M})$ up to unique isomorphism as a cokernel of a map that is independent of the cohomology theory, and this is true for every $\mathcal{O}$-module $\mathcal{M}$. So it is also true that $H^1(X, \mathcal{M}^1)$ is determined uniquely. This being so, $H^2(X, \mathcal{M})$ is determined uniquely for every $\mathcal{M}$, by the map (7.4.6), with $q = 1$. The isomorphisms (7.4.6) determine the rest of the cohomology up to unique isomorphism by induction on q.

We verify the uniqueness of the coboundary maps δ^q in the cohomology sequence (7.2.3) at the end of the section.

Existence of cohomology: We repeat the construction of $\mathcal{R}^0_{\mathcal{M}}$ with $\mathcal{M}^1$. Let $\mathcal{R}^1_{\mathcal{M}} = \mathcal{R}^0_{\mathcal{M}^1}$ $(= j_* j^* \mathcal{M}^1)$. The injective map $\mathcal{M}^1 \to \mathcal{R}^0_{\mathcal{M}^1}$, combined with the exact sequence (7.4.1) **(ii)**, gives us an exact sequence

RMone (7.4.7)
$$0 \to \mathcal{M} \to \mathcal{R}^0_{\mathcal{M}} \to \mathcal{R}^1_{\mathcal{M}}.$$

Setting $\mathcal{M}^2 = \mathcal{R}^1_{\mathcal{M}}/\mathcal{M}^1$ and $j_* j^* \mathcal{M}^2 = \mathcal{R}^2_{\mathcal{M}}$, etc., we construct modules $\mathcal{R}^n = \mathcal{R}^n_{\mathcal{M}}$ forming an exact sequence

RM (7.4.8)
$$0 \to \mathcal{M} \to \mathcal{R}^0 \to \mathcal{R}^1 \to \mathcal{R}^2 \to \cdots.$$

The next lemma follows by induction from Lemmas 7.4.1 and 7.4.2.

Rcmexact **7.4.9. Lemma.** *Given a short exact sequence* $0 \to \mathcal{M} \to \mathcal{N} \to \mathcal{P} \to 0$*, the sequences*

$$0 \to \mathcal{R}^n_{\mathcal{M}} \to \mathcal{R}^n_{\mathcal{N}} \to \mathcal{R}^n_{\mathcal{P}} \to 0$$

are exact for every n*, and so are the sequences of global sections*

$$0 \to \mathcal{R}^n_{\mathcal{M}}(X) \to \mathcal{R}^n_{\mathcal{N}}(X) \to \mathcal{R}^n_{\mathcal{P}}(X) \to 0.$$

If H^q *is a cohomology theory, then* $H^q(X, \mathcal{R}^n_{\mathcal{M}}) = 0$ *for all* $\mathcal{M}$ *and all* $q > 0$. □

An exact sequence such as (7.4.8) is called a *resolution* of $\mathcal{M}$, and because $H^q(X, \mathcal{R}^n) = 0$ for $q > 0$, it is an *acyclic resolution* of $\mathcal{M}$.

Continuing with the proof of existence, we consider the sequence of $\mathcal{O}$-modules $\mathcal{R}^\bullet_{\mathcal{M}}$ that is obtained by omitting the first term from (7.4.8):

Rsequence (7.4.10) $$0 \to \mathcal{R}^0_{\mathcal{M}} \to \mathcal{R}^1_{\mathcal{M}} \to \mathcal{R}^2_{\mathcal{M}} \to \cdots$$

and the sequence $\mathcal{R}^\bullet_{\mathcal{M}}(X)$ of its global sections:

RMX (7.4.11) $$0 \to \mathcal{R}^0_{\mathcal{M}}(X) \to \mathcal{R}^1_{\mathcal{M}}(X) \to \mathcal{R}^2_{\mathcal{M}}(X) \cdots .$$

We could also write the sequence of global sections as

HXR (7.4.12) $$0 \to H^0(X, \mathcal{R}^0_{\mathcal{M}}) \to H^0(X, \mathcal{R}^1_{\mathcal{M}}) \to H^0(X, \mathcal{R}^2_{\mathcal{M}}) \cdots .$$

The sequence (7.4.10) is exact except at $\mathcal{R}^0_{\mathcal{M}}$, but because the global section functor is only left exact, the sequence (7.4.11) of its global sections needn't be exact anywhere. However, it is a complex because $\mathcal{R}^\bullet_{\mathcal{M}}$ is a complex. The composition of two adjacent maps is zero.

Recall that the cohomology of a complex $V^\bullet$ of vector spaces is defined to be $\mathbf{h}^q(V^\bullet) = (\ker\, d^q)/(\mathrm{im}\, d^{q-1})$, and that $\{\mathbf{h}^q\}$ is a cohomological functor on complexes.

definecoh **7.4.13. Definition.** The cohomology of an $\mathcal{O}$-module $\mathcal{M}$ is

defH (7.4.14) $$H^q(X, \mathcal{M}) = \mathbf{h}^q(\mathcal{R}^\bullet_{\mathcal{M}}(X)).$$

Thus if we denote the maps in the complex (7.4.11) by d^q:

$$0 \to \mathcal{R}^0_{\mathcal{M}}(X) \xrightarrow{d^0} \mathcal{R}^1_{\mathcal{M}}(X) \xrightarrow{d^1} \mathcal{R}^2_{\mathcal{M}}(X) \to \cdots$$

then $H^q(X, \mathcal{M}) = (\ker\, d^q)/(\mathrm{im}\, d^{q-1})$.

affinecohzero **7.4.15. Lemma.** *Let* X *be an affine variety. Then with cohomology defined as in (7.4.14),* $H^q(X, \mathcal{M}) = 0$ *for all* $\mathcal{O}$*-modules* $\mathcal{M}$ *and all* $q > 0$.

proof. When X is affine, the sequence of global sections of the exact sequence (7.4.8) is exact. □

To show that our definition gives the unique cohomology theory, we verify the characteristic properties. Since (7.4.8) is exact and since the global section functor is left exact, $\mathcal{M}(X)$ is the kernel of the map $\mathcal{R}^0(X) \to \mathcal{R}^1(X)$, and this kernel is $\mathbf{h}^0(\mathcal{R}^\bullet_{\mathcal{M}}(X))$. So the first property, $H^0(X, \mathcal{M}) = \mathcal{M}(X)$, is verified.

To show that we obtain a cohomological functor, we apply Lemma 7.4.9, to conclude that for a short exact sequence $0 \to \mathcal{M} \to \mathcal{N} \to \mathcal{P} \to 0$, the global sections

crtwo (7.4.16) $$0 \to \mathcal{R}^\bullet_{\mathcal{M}}(X) \to \mathcal{R}^\bullet_{\mathcal{N}}(X) \to \mathcal{R}^\bullet_{\mathcal{P}}(X) \to 0,$$

form an exact sequence of complexes. Since cohomology of complexes is a cohomological functor, so is $H^q(X, \cdot)$.

We need a definition before verifying the third characteristic property. An *affine morphism* $Y \xrightarrow{f} X$ is a morphism of varieties with the property that the inverse image $f^{-1}(U)$ of any affine open subset of X is an affine open subset of Y.

affinemorph **7.4.17. Lemma.** *The following are examples of affine morphisms:*

- *the inclusion of an affine open subset Y into X,*
- *the inclusion of a closed subvariety Y into X.*
- *an integral morphism.* □

Let $Y \xrightarrow{f} X$ be an affine morphism, let $\mathbb{U} = \{U^\nu\}$ be our chosen affine covering of X, and let $V^\nu = f^{-1}U^\nu$. Then $\mathbb{V} = \{V^\nu\}$ is an affine covering of Y. Let $\mathbb{U} \xrightarrow{j} X$ and $\mathbb{V} \xrightarrow{j'} Y$ denote the obvious families of maps. Let $H^q(X, \cdot\,)$ be the cohomology defined in (7.4.13), and let $H^q(Y, \cdot\,)$ denote the cohomology defined in the analogous way, using the covering $\mathbb{V}$ of Y.

affinedirec-timage **7.4.18. Lemma.** *Let $Y \xrightarrow{f} X$ be an affine morphism, and let $\mathcal{N}$ be an $\mathcal{O}_Y$-module. With cohomology defined as above, $H^q(X, f_*\mathcal{N}) = H^q(Y, \mathcal{N})$.*

proof. To compute the cohomology of $f_*\mathcal{N}$ on X, we substitute $\mathcal{M} = f_*\mathcal{N}$ into Definition 7.4.13:

$$H^q(X, f_*\mathcal{N}) = \mathbf{h}^q(\mathcal{R}^\bullet_{f_*\mathcal{N}}(X)).$$

To compute the cohomology of $\mathcal{N}$ on Y, we let $\mathcal{R}'^0_{\mathcal{N}} = j'_*j'^*\mathcal{N}$, and we continue, to construct a resolution $\mathcal{R}'^\bullet_{\mathcal{N}}(Y) \;=\; 0 \to \mathcal{N} \to \mathcal{R}'^0_{\mathcal{N}} \to \mathcal{R}'^1_{\mathcal{N}} \to \cdots$ and the complex of its global sections $\mathcal{R}'^\bullet_{\mathcal{N}}(Y)$. Then

$$H^q(Y, \mathcal{N}) = \mathbf{h}^q(\mathcal{R}'^\bullet_{\mathcal{N}}(Y)).$$

It suffices to show that $\mathcal{R}^\bullet_{f_*\mathcal{N}}(X) = \mathcal{R}'^\bullet_{\mathcal{N}}(Y)$. We refer to Lemma 7.4.1:

$$\mathcal{R}^\bullet_{f_*\mathcal{N}}(X) = \prod [f_*\mathcal{N}](U^\nu) \;\overset{def}{=}\; \prod \mathcal{N}(V^\nu) = \mathcal{R}'^\bullet_{\mathcal{N}}(Y).$$ □

The third characteristic property of cohomology follows from Lemma 7.4.15. If f is the inclusion of an affine open subset Y into X and $\mathcal{N}$ is an $\mathcal{O}_Y$-module, then f is an affine morphism, so $H^q(X, f_*\mathcal{N}) = H^q(Y, \mathcal{N})$. And since Y is an affine variety, $H^q(Y, \mathcal{N}) = 0$ for all $q > 0$. □

proof of uniqueness, continued. We have constructed a cohomology theory $\{H^q\}$ complete with coboundary maps δ^q, and we have shown that the functors H^q are unique. We haven't yet shown that the coboundary maps in (7.2.3) are unique. To make it clear that there is something to show, we note that the cohomology sequence (7.2.3) remains exact when some of the coboundary maps δ^q are multiplied by -1. Why can't we define a new collection of coboundary maps by changing some signs? The reason is that we used the coboundary maps δ^q in (7.4.5) and (7.4.6) to identify $H^q(X, \mathcal{M})$. Having done that, we aren't allowed to change δ^q for the particular short exact sequences (7.4.1) **(ii)**. We show that the coboundary maps for those particular sequences determine the coboundary maps for every short exact sequence of $\mathcal{O}$-modules

A (7.4.19) $$(A) \qquad 0 \to \mathcal{M} \longrightarrow \mathcal{N} \longrightarrow \mathcal{P} \to 0$$

To show that the coboundaries for the sequence (A) are determined uniquely, we relate that sequence to a sequence for which the coboundary maps are fixed:

B (7.4.20) $$(B) \qquad 0 \to \mathcal{M} \longrightarrow \mathcal{R}^0_{\mathcal{M}} \longrightarrow \mathcal{M}_1 \to 0$$

We map (B) to a third short exact sequence (C), obtaining the diagram below, whose rows and columns are short exact sequences. (We have suppressed zeros on the ends.) The map ψ is the composition of the injective maps $\mathcal{M} \to \mathcal{R}^0_{\mathcal{M}} \to \mathcal{R}^0_{\mathcal{N}}$ and $\mathcal{Q}$ is the cokernel of ψ. The exactness of all rows and columns follows from the Snake Lemma.

BtoC (7.4.21)

$$\begin{array}{cccccc}
(B) & \mathcal{M} & \longrightarrow & \mathcal{R}^0_{\mathcal{M}} & \longrightarrow & \mathcal{M}_1 \\
& \| & & \downarrow & & \downarrow \\
(C) & \mathcal{M} & \xrightarrow{\psi} & \mathcal{R}^0_{\mathcal{N}} & \longrightarrow & \mathcal{Q} \\
. & \downarrow & & \downarrow & & \downarrow \\
& 0 & \longrightarrow & \mathcal{R}^0_{\mathcal{P}} & = & \mathcal{R}^0_{\mathcal{P}}
\end{array}$$

The sequence (A) can also be mapped to the sequence (C):

firstdiagram (7.4.22)

$$\begin{array}{cccccc} (A) & \mathcal{M} & \longrightarrow & \mathcal{N} & \longrightarrow & \mathcal{P} \\ & \Big\| & & \Big\downarrow & & \Big\downarrow \\ (C) & \mathcal{M} & \xrightarrow{\psi} & \mathcal{R}^0_{\mathcal{N}} & \longrightarrow & \mathcal{Q} \end{array}$$

and this gives us a diagram of coboundary maps

second (7.4.23)

$$\begin{array}{cccc} (A) & H^q(X, \mathcal{P}) & \xrightarrow{\delta^q_A} & H^{q+1}(X, \mathcal{M}) \\ & \Big\downarrow & & \Big\| \\ (C) & H^q(X, \mathcal{Q}) & \xrightarrow{\delta^q_C} & H^{q+1}(X, \mathcal{M}) \end{array}$$

This diagram determines the coboundary maps for (A) in terms of the coboundary maps for the sequence (C). So when we show that the coboundary maps for (C) are determined by the unique coboundary maps for (B), the proof will be complete.

We go back to the diagram 7.4.21. That diagram gives us a diagram of coboundary maps

twosquares (7.4.24)

$$\begin{array}{cccc} (B) & H^q(X, \mathcal{M}_1) & \xrightarrow{\delta^q_B} & H^{q+1}(X, \mathcal{M}) \\ & \Big\downarrow & & \Big\| \\ (C) & H^q(X, \mathcal{Q}) & \xrightarrow{\delta^q_C} & H^{q+1}(X, \mathcal{M}) \end{array}$$

When $q > 0$, δ^q_C and δ^q_B are bijective because the cohomology of $\mathcal{R}^0_{\mathcal{M}}$ and $\mathcal{R}^0_{\mathcal{N}}$ is zero in dimensions q and $q+1$. Then $H^q(X, \mathcal{M}_1) \approx H^q(X, \mathcal{Q})$, so δ^q_C is uniquely determined by the coboundary map δ^q_B, and so is δ^q_A.

Unfortunately, we have to look more closely to settle the case $q = 0$. For $q = 0$, the relevant part of the cohomology diagram associated to this diagram is

$$\begin{array}{cccccc} (B) & H^0(X, \mathcal{R}^0_{\mathcal{M}}) & \longrightarrow & H^0(X, \mathcal{M}_1) & \xrightarrow{\delta^0_B} & H^1(X, \mathcal{M}) \\ & \Big\downarrow & & \Big\downarrow{\scriptstyle \alpha} & & \Big\| \\ (C) & H^0(X, \mathcal{R}^0_{\mathcal{N}}) & \xrightarrow{\beta} & H^0(X, \mathcal{Q}) & \xrightarrow{\delta^0_C} & H^1(X, \mathcal{M}) \\ & \Big\downarrow{\scriptstyle \gamma} & & \Big\downarrow & & \\ & H^0(X, \mathcal{R}^0_{\mathcal{P}}) & = & H^0(X, \mathcal{R}^0_{\mathcal{P}}) & & \end{array}$$

In this diagram, the rows and columns are exact, and with the possible exception of δ^0_C, the maps are determined uniquely. The map δ^0_C is zero on the image of β and it is determined by δ^0_B on the image of α. To show that δ^0_C is determined, it suffices to show that the images of α and β together span $H^0(X, \mathcal{Q})$. This follows from the fact that, because $H^1(X, \mathcal{R}^0_{\mathcal{M}}) = 0$, the map γ is surjective. Thus δ^0_C is determined uniqely by δ^0_B, and so is δ^0_A. $\square$

cohprojsp 7.5 Computations

In this section, we determine the cohomology of the twisting sheaves $\mathcal{O}(d)$ on $\mathbb{P}^n$. We have described the global sections of $\mathcal{O}(d)$ before: For $d \geq 0$, $H^0(X, \mathcal{O}(d))$ is the space of homogeneous polynomials of degree d in the variables $x_0, ..., x_n$. Its dimension is $\binom{d+n}{n}$ (see(6.2.11)). Theorem 7.5.4 below tells us that the cohomology $H^q(X, \mathcal{O}(d))$ is zero for most values of q. This is a great help for computing the cohomology of other modules.

Lemma 7.4.15 about vanishing of cohomology on an affine variety and Lemma 7.4.18 about the direct image via an affine morphism were stated using particular affine coverings. Since we know now that cohomology is unique, those particular coverings are irrelevant. So, though it may not be necessary, we restate the lemmas here as a corollary:

affinecohzerotwo **7.5.1. Corollary.** **(i)** *If $\mathcal{M}$ is a $\mathcal{O}$-module on an affine variety X, then $H^q(X,\mathcal{M}) = 0$ for all $q > 0$.*

(ii) *Let $Y \xrightarrow{f} X$ be an affine morphism, and let $\mathcal{N}$ be an $\mathcal{O}_Y$-module. Then $H^q(X, f_*\mathcal{N})$ and $H^q(Y,\mathcal{N})$ are isomorphic. If Y is an affine variety, then $H^q(X, f_*\mathcal{N}) = 0$ for all $q > 0$.* □

Note that this corollary applies when f is the inclusion of a closed subvariety Y into X.

The next lemma is a simple consequence of the definition of direct limit (see **6.5.13**).

cohlimit **7.5.2. Lemma.** *Cohomology is compatible with limits of directed sets of $\mathcal{O}$-modules: $H^q(X, \varinjlim \mathcal{M}_\bullet) \approx \varinjlim H^q(X, \mathcal{M}_\bullet)$ for all q.*

proof. The direct and inverse image functors and the global section functor are all compatible with $\varinjlim$, and therefore the complex used to compute cohomology of $\varinjlim \mathcal{M}_\bullet$ is $\mathcal{R}^\bullet_{\varinjlim \mathcal{M}_\bullet}(X) \approx \varinjlim [\mathcal{R}^\bullet_{\mathcal{M}_\bullet}](X)$. Since $\varinjlim$ is an exact operation, $\mathbf{h}^q(\mathcal{R}^\bullet_{\varinjlim \mathcal{M}_\bullet}(X)) \approx \varinjlim [\mathbf{h}^q(\mathcal{R}^\bullet_{\mathcal{M}_\bullet})](X)$. □

We compute some cohomology on the projective space $X = \mathbb{P}^d$ now. Let $\mathcal{M}$ be a finite $\mathcal{O}$-module. As before (**6.6.6**), the twisting sheaves $\mathcal{O}(n)$ and the twists $\mathcal{M}(n)$ of a module $\mathcal{M}$ can be assembled into directed sets

$$\mathcal{O} \xrightarrow{x_0} \mathcal{O}(1) \xrightarrow{x_0} \mathcal{O}(2) \xrightarrow{x_0} \cdots,$$

and

$$\mathcal{M} \xrightarrow{x_0} \mathcal{M}(1) \xrightarrow{x_0} \mathcal{M}(2) \xrightarrow{x_0} \cdots$$

in which each map is multiplication by x_0. Moreover, $\varinjlim_{x_0} \mathcal{O}(n) \approx j_*j^*\mathcal{O}_X$ and $\varinjlim_{x_0} \mathcal{M}(n) \approx j_*j^*\mathcal{M}$ (6.6.11).

Onsequencetwo **7.5.3. Corollary.** *With notation as above, $\varinjlim_{x_0} H^q(X, \mathcal{O}(n)) = 0$, and $\varinjlim_{x_0} H^q(X, \mathcal{M}(n)) = 0$ for all $q > 0$.* □

cohOd **7.5.4. Theorem.** *Let X denote the projective space $\mathbb{P}^n$.*
(i) *For $d \geq 0$, $H^q(X, \mathcal{O}(d)) = 0$ if $q \neq 0$.*
(ii) *For $r > 0$, $H^q(X, \mathcal{O}(-r)) = 0$ if $q \neq n$. The dimension of $H^n(X, \mathcal{O}(-r))$ is $\binom{r-1}{n}$.*

For instance, let X be the projective plane $\mathbb{P}^2$. If $r > 0$, then $\dim H^2(X, \mathcal{O}(-r)) = r-1$ and $H^q(X, \mathcal{O}(-r)) = 0$ for $q \neq 2$. Let $C \xrightarrow{i} X$ be the inclusion of a plane curve of degree r. As we saw in the last chapter (**6.2.14**), the ideal $\mathcal{I}$ of C is isomorphic to the twisting sheaf $\mathcal{O}_X(-r)$, and one has an exact sequence

$$0 \to \mathcal{O}_X(-r) \to \mathcal{O}_X \to i_*\mathcal{O}_C \to 0$$

Instead of displaying the cohomology sequence, we form a table that shows the dimensions of the cohomology $H^q(X, \mathcal{M})$ below the module $\mathcal{M}$. All of the cohomology groups in dimension greater than 2 are zero.

cohdims (7.5.5)

	$\mathcal{O}_X(-r)$	$\mathcal{O}_X$	$i_*\mathcal{O}_C$
H^0 :	0	1	1
H^1 :	0	0	$\binom{r-1}{2}$
H^2 :	$\binom{r-1}{2}$	0	0

The theorem determines the first two columns, and since the cohomology sequence is exact, the last column is determined. Thus $\dim H^0(C, \mathcal{O}_C) = 1$ and $\dim H^1(C, \mathcal{O}_C)$ is the binomial coefficient $\binom{r-1}{2}$. The dimension of $H^1(C, \mathcal{O}_C)$ is called the *arithmetic genus* of C, and is usually denoted by p_a.

In the next chapter we will see that the arithemtic genus p_a of a *smooth* curve is equal to its topological genus, which is denoted by g. It is interesting that the arithmetic genus of a curve of degree d is equal to $\binom{r-1}{2}$ also when C is singular, even when C is a reducible curve.

cohplanecurve **7.5.6. Corollary.** *Let C be a plane curve of degree r. Then $H^0(C, \mathcal{O}_C) = \mathbb{C}$, the dimension of $H^1(C, \mathcal{O}_C)$ is the arithmetic genus $p_1 = \binom{r-1}{2}$ of C, and $H^q = 0$ if $q \neq 0, 1$.* □

The fact that the only global sections of $\mathcal{O}_Y$ are the constants says that the only rational functions that are regular everywhere on Y are the constants. This reflects the fact that a plane curve is connected, though it isn't a proof of that fact.

proof of Theorem 7.5.4. **(i)** (*the case* $d \geq 0$) Let Y be the hyperplane at infinity in $\mathbb{P}^n$ By induction on n, we may assume that the theorem has been proved for Y, which is a projective space of dimension $n-1$. We consider the exact sequence

Od (7.5.7) $$0 \to \mathcal{O}(d{-}1) \xrightarrow{x_0} \mathcal{O}(d) \to i_*\mathcal{O}_Y(d) \to 0.$$

obtained by twisting the sequence (**??**), where i is the inclusion of Y into X. Because the inclusion of Y into X is an affine morphism, $H^q(X, i_*\mathcal{O}_Y(d)) \approx H^q(Y, \mathcal{O}(d))$ (7.4.18). Remembering that $Y = \mathbb{P}^{n-1}$, one sees that every global section of $\mathcal{O}_Y(d)$ is the image of a global section of $\mathcal{O}(d)$, and therefore that the sequence of global sections

$$0 \to H^0(X, \mathcal{O}(d{-}1)) \xrightarrow{x_0} H^0(X, \mathcal{O}_X(d)) \to H^0(Y, \mathcal{O}_Y(d)) \to 0$$

is exact. By induction on n, $H^q(Y, \mathcal{O}_Y(d)) = 0$ for $d \geq 0$ and $q > 0$. Therefore the cohomology sequence provides bijections $H^q(X, \mathcal{O}_X(d{-}1)) \approx H^q(X, \mathcal{O}_X(d))$ for all $q > 0$. Since the limit is zero (7.5.3), $H^q(X, \mathcal{O}_X(d)) = 0$ for $q > 0$.

(ii) (*the case* $d < 0$) We substitute $d = -r$ into (7.5.7):

Or (7.5.8) $$0 \to \mathcal{O}_X(-r{-}1) \xrightarrow{x_0} \mathcal{O}_X(-r) \to i_*\mathcal{O}_Y(-r) \to 0$$

The cohomology sequence associated to this sequence, with $r = 1$, shows that $H^q(X, \mathcal{O}_X(-1)) = 0$ for every q. Then, using induction on $r > 1$, one sees that $H^q(X, \mathcal{O}_X(-r)) = 0$ if $q \neq n$, and that the cohomology sequence becomes an exact sequence

$$0 \to H^{n-1}(Y, \mathcal{O}_Y(-r)) \to H^n(X, \mathcal{O}_X(-r-1)) \to H^n(X, \mathcal{O}_X(-r)) \to 0$$

The result now follows from Pascal's Rule

$$\binom{r+1}{n} = \binom{r}{n} + \binom{r}{n-1}$$

□

7.6 Finiteness

findim

In this section, we show that when X is projective variety, the cohomology $H^q(X, \mathcal{M})$ of a finite $\mathcal{O}_X$-module $\mathcal{M}$ is a finite dimensional vector space.

We first define support for a finite module over a ring A.

support **(7.6.1) support of an $\mathcal{O}$-module**

The *annihilator* I of an A-module M is the set of elements α of A such that $\alpha M = 0$. The annihilator is an ideal of A.

defsupp **7.6.2. Proposition.** *Let A be a finite-type domain, let $X = \operatorname{Spec} A$, let M be a finite A-module, and let I be the annihilator of M. The following are descriptions of the same closed subset S of X. This subset is the* **support** *of M:*

(a) *$S = V_X(I)$ is the zero locus of I in X.*

(b) *S is the set of points p of X such that I is contained in the maximal ideal $\mathfrak{m}_p$.*

(c) *S is the set of points p of X such that $\mathfrak{m}_p M < M$, or that $M/\mathfrak{m}_p M \neq 0$.*

proof. **(b)** is the definition of $V_X(I)$, so it is equivalent with **(a)**. We show that **(b)** and **(c)** are equivalent. If $\mathfrak{m}_p M = M$, the Nakayama Lemma tells us that there is an element z in $\mathfrak{m}_p$ such that $1 - z$ annihilates M. Then $1 - z$ isn't in $\mathfrak{m}_p$, so $I \not\subset \mathfrak{m}_p$. Conversely, suppose that $I \not\subset \mathfrak{m}_p$. Let α be an element of I that isn't in $\mathfrak{m}_p$. We evaluate the function α at p. The value $\alpha(p)$ will be a nonzero complex number c, and then $z = c - \alpha$ is in $\mathfrak{m}_p$. Since α annihilates M and $c \neq 0$, $zM = cM = M$. Therefore $\mathfrak{m}_p M = M$. □

localize-support **7.6.3. Lemma.** *With notation as in the previous proposition, let s be a nonzero element of A. The annihilator of the localized module M_s is the localized ideal I_s, and the support of M_s is the set $S \cap X_s$ of points of X_s in S.* □

The lemma allows us to extend the concept of annihilator and support to finite $\mathcal{O}$-modules on a variety X by the standard procedure: looking on affine open sets. The support of a finite $\mathcal{O}_X$-module is a closed subset of X.

multker-nelzero **7.6.4. Lemma.** **(i)** *Let M be a finite module over a noetherian domain A, and let α be an element of A. For all but finitely many complex numbers c, multiplication by $c+\alpha$ defines an injective map $M \to M$.*

(ii) *Let $\mathcal{M}$ be a finite $\mathcal{O}$-module on projective space $X = \mathbb{P}^n$. and let $z = c_0x_0 + \cdots + c_nx_n$ be a generic homogeneous linear polynomial in the coordinate variables x. The kernel of multiplication by z: $\mathcal{M}(d-1) \xrightarrow{z} \mathcal{M}(d)$ is zero.*

proof. **(i)** Let $\alpha_c = c + \alpha$, let N_c denote the kernel of multiplication by α_c on M, and let S denote the sum of these submodules. Since A is noetherian and M is a finite module, S is a finite module. The point is that the submodules N_c are independent, and therefore that their sum S is the direct sum $\bigoplus N_c$ The direct sum of infinitely many nonzero modules isn't a finite module, so almost all summands must be zero.

We verify that the modules N_c are independent. Suppose $c_1, ..., c_k$ are distinct scalars and that there is a relation $n_1 + \cdots + n_k = 0$, with nonzero n_i in N_{c_i}. Multiplying by α_{c_1} annihilates n_1 but it not $n_2, ..., n_k$. This gives us a relation with fewer terms.

(ii) It suffices to show that the restriction of the map to the standard affine open sets is injective. On the standard affine opens, it follows from **(i)**.

We are now ready for the finiteness theorem.

cohfinite **7.6.5. Theorem.** *Let X be a projective variety, and let $\mathcal{M}$ be a finite $\mathcal{O}_X$-module, and let $\mathcal{M}(r)$ denote its twist.*

(i) *If the support of $\mathcal{M}$ has dimension k, then $H^q(X, \mathcal{M}) = 0$ for all $q > k$. If X has dimension n, then $H^q(X, \mathcal{M}) = 0$ for all $q > n$.*

(ii) *the cohomology $H^q(X, \mathcal{M})$ is a finite-dimensional vector space for every q, and*

(iii) *for sufficiently large r and for every $q > 0$, $H^q(X, \mathcal{M}(r)) = 0$.*

Theorem 7.6.5 allows us to define the *Euler characteristic* $\chi(\mathcal{M})$ of a finite $\mathcal{O}$-module $\mathcal{M}$ on projective variety X, as the alternating sum of the dimensions of the cohomology:

chi (7.6.6)
$$\chi(\mathcal{M}) = \sum (-1)^q \dim\, H^q(X, \mathcal{M}).$$

chiofse-quence **7.6.7. Proposition.** *Let $0 \to \mathcal{M} \to \mathcal{N} \to \mathcal{P} \to 0$ be a short exact sequence of finite $\mathcal{O}$-modules on a projective variety. Then $\chi(\mathcal{M}) - \chi(\mathcal{N}) + \chi(\mathcal{P}) = 0$.*

proof. Since the cohomology sequence associated to the given short exact sequence is exact, the alternating sum of the dimensions of its terms is zero (7.1.8). On the other hand, the alternating sum is also equal to $\chi(\mathcal{M}) - \chi(\mathcal{N}) + \chi(\mathcal{P})$. □

descind **7.6.8. Note.** **(i)** Suppose that X is a projective variety of dimension n. Then as is suggested by Theorem 7.6.5**(i)**, the *cohomological dimension* for $\mathcal{O}_X$-modules, the highest dimension in which cohomology can be nonzero, is equal to its algebraic dimension n. On the other hand, in the classical topology, the constant coefficient cohomology $H^{2n}_{class}(X, \mathbb{Z})$ will be nonzero. The cohomological dimension of X in the classical topology is its topological dimension, which is $2n$. The Zariski topology is too coarse to use for cohomology with constant coefficients. In the Zariski topology, $H^q(X, \mathbb{Z})$ is zero for every $q > 0$.

(ii) It isn't easy to prove directly that the space $H^0(X, \mathcal{M})$ of global sections of a finite $\mathcal{O}$-module on a projective variety is finite-dimensional. The proof of part **(ii)** of Theorem 7.6.5 uses *descending* induction on q. □

The proof of Theorem 7.6.5 is based on the cohomology of the twisting sheaves, the vanishing of the limit (7.5.3), and on two exact sequences. The first sequence is one that was introduced before. Lemma 7.6.4 tells us that if coordinates are in general position, there will be an exact sequence

threeones (7.6.9) $$0 \to \mathcal{M}(d{-}1) \xrightarrow{x_0 \otimes 1} \mathcal{M}(d) \to \overline{\mathcal{M}}(d) \to 0$$

where $\overline{\mathcal{M}} = \mathcal{M}/\mathcal{M}(-1)$. If S is the support of $\mathcal{M}$, the support of $\overline{\mathcal{M}}$ will be $S \cap H$, where H is the hyperplane $x_0 = 0$. So if $\dim S = k$, the support of $\overline{\mathcal{M}}$ willhave have dimension $k{-}1$. This will allow us to use induction on k and d.

Next, $\mathcal{M}(r)$ is generated by global sections if r is sufficiently large (**??**). Choosing generators gives us a surjective map $\mathcal{O}^m \to \mathcal{M}(r)$. Let $\mathcal{N}$ be the kernel of this map. We twist, obtaining short exact sequences

threetwos (7.6.10) $$0 \to \mathcal{N}(n) \to \mathcal{O}(n)^m \to \mathcal{M}(n{+}r) \to 0$$

for each n. This is useful because we know the cohomology of $\mathcal{O}(n)$.

proof of Theorem 7.6.5. **(ii)** (*vanishing in large dimension*) We inspect the sequence (7.6.9). By induction on the dimension k of the support of the finite $\mathcal{O}$-module $\mathcal{M}$, we may assume that $H^q(X, \overline{\mathcal{M}}(n)) = 0$ for all $q > k{-}1$ and all n. (When $k = 0$, $\overline{\mathcal{M}} = 0$ and then $H^q(X, \overline{\mathcal{M}}(n)) = 0$ for all q.) The cohomology sequence associated to (7.6.9) is

XX (7.6.11) $$\to H^{q-1}(X, \overline{\mathcal{M}}(n)) \xrightarrow{\delta^{q-1}} H^q(X, \mathcal{M}(n{-}1)) \to H^q(X, \mathcal{M}(n)) \to H^q(X, \overline{\mathcal{M}}(n)) \xrightarrow{\delta^q}$$

If $q > k$, the terms on the left and right are zero, and therefore the map

$$H^q(X, \mathcal{M}(n{-}1)) \to H^q(X, \mathcal{M}(n))$$

is an isomorphism. According to (7.5.3), the limit is zero, so $H^q(X, \mathcal{M}(n)) = 0$ for all n. In particular, $H^q(X, \mathcal{M}) = 0$.

proof of Theorem 7.6.5 **(iii)** (*vanishing for a large twist*): The cohomology sequence associated to the sequence (7.6.10) is

YY (7.6.12) $$\to H^q(X, \mathcal{O}(d))^m \to H^q(X, \mathcal{M}(d{+}r)) \xrightarrow{\delta^q} H^{q+1}(X, \mathcal{N}(d)) \to H^{q+1}(X, \mathcal{O}(d))^m \to$$

We have proved that $H^q(X, \mathcal{O}(d)) = 0$ when $d \geq 0$ and $q > 0$, and then δ^q is bijective. We set $q = n$, where n is the dimension of X. Then $H^{n+1}(X, \mathcal{N}(d)) = 0$, and therefore $H^n(X, \mathcal{M}(d{+}r)) = 0$. The particular integer $d + r$ won't be useful. What has been shown is that

threethrees (7.6.13) $$H^n(X, \mathcal{M}(d)) = 0, \quad \text{if } d \text{ is sufficiently large,}$$

and this is true for every finite $\mathcal{O}$-module $\mathcal{M}$, though how large d must be depends on $\mathcal{M}$.

Let k be a positive integer. Suppose that *for all finite $\mathcal{O}$-modules $\mathcal{M}$*, there is an integer d_0 such that $H^{k+1}(X, \mathcal{M}(d)) = 0$ if $d \geq d_0$. We substitute $q = k$ into (7.6.12), obtaining bijections

$$H^k(X, \mathcal{M}(d{+}r)) \xrightarrow{\delta^k} H^{k+1}(X, \mathcal{N}(d)),$$

and we apply our hypothesis to the case that $\mathcal{M} = \mathcal{N}$ to conclude that $H^{k+1}(X, \mathcal{N}(d)) = 0$ for large d, and therefore $H^k(X, \mathcal{M}(d{+}r)) = 0$. This tells us that $H^k(X, \mathcal{M}(d)) = 0$ for large d. Descending induction shows that $H^k(X, \mathcal{M}(d)) = 0$ for large d and all positive k, which completes the proof. □

Theorem 7.6.5 **(i)**, (*finiteness of cohomology*): We go back to the sequence (7.6.9) and its cohomology sequence (7.6.11). Induction on the dimension of the support of $\mathcal{M}$ allows us to assume that the cohomology of $\overline{\mathcal{M}}(n)$ is finite-dimensional for all q. We conclude from the sequence (7.6.11) that either $H^q(X, \mathcal{M}(n-1))$ and $H^q(X, \mathcal{M}(n))$ are both finite-dimensional, or else they are both infinite-dimensional.

If $q > 0$, then $H^q(X, \mathcal{M}(n)) = 0$ for large n. Since the zero space is finite-dimensional, we can use the sequence together with descending induction on n to conclude that $H^q(X, \mathcal{M}(n))$ is finite-dimensional for all finite modules $\mathcal{M}$ and all n, and therefore that $H^q(X, \mathcal{M})$ is finite-dimensional.

To prove that $H^0(X, \mathcal{M})$ is finite-dimensional, we twist the sequence (7.6.10) by $-n - r$:

$$0 \to \mathcal{N}(-r) \to \mathcal{O}(-r)^m \to \mathcal{M} \to 0$$

The corresponding cohomology sequence is

$$0 \to H^0(X,\mathcal{N}(-r)) \to H^0(X,\mathcal{O}(-r))^m \to H^0(X,\mathcal{M}) \xrightarrow{\delta^0} H^1(X,\mathcal{N}(-r)) \to \cdots.$$

We have shown that the second and fourth terms here are finite-dimensional, and it follows that the third term is finite-dimensional. This completes the proof.

Notice that the finiteness of H^0 comes out only at the end. □

cohhyper

7.7 Cohomology of Hypersurfaces

Let Y be a hypersurface in $X = \mathbb{P}^d$, with $d > 0$, the locus of zeros of an irreducible homogeneous polynomial f of degree k. The ideal of Y is isomorphic to $\mathcal{O}_X(-d)$ (see 6.2.15). So there is an exact sequence

OY (7.7.1)
$$0 \to \mathcal{O}_X(-k) \xrightarrow{f} \mathcal{O}_X \to i_*\mathcal{O}_Y \to 0$$

Since we know the cohomology of twists of the structure sheaf on X (7.5.4), and since $H^q(X, i_*\mathcal{O}_Y) \approx H^q(Y, \mathcal{O}_Y)$, we can use this sequence to compute the cohomology of $\mathcal{O}_Y$.

cohhyper-surface **7.7.2. Corollary.** *Let Y be a hypersurface of degree k in $\mathbb{P}^d$, with $d \geq 2$. Then $H^0(Y, \mathcal{O}_Y) = \mathbb{C}$, The dimension of $H^{d-1}(Y, \mathcal{O}_Y)$ is $\binom{k-1}{d}$, and $H^q(Y, \mathcal{O}_Y)$ is zero for all other q.* □

For example, when $d = 2$, $H^1(Y, \mathcal{O}_Y)$ is isomorphic to $H^2(X, \mathcal{O}_X(-k))$. We have noted this before (see 7.5.5).

paone **7.7.3. Corollary.** *Let Y be the surface in projective 3-space defined by an irreducible polynomial of degree k. Then $H^0(Y, \mathcal{O}_Y) = \mathbb{C}$, the dimension of $H^2(Y, \mathcal{O}_Y)$ is $\binom{k-1}{3}$, and $H^q = 0$ if $q \neq 0, 2$.* □

For a projective surface, the dimensions of $H^1(Y, \mathcal{O}_Y)$ and $H^2(Y, \mathcal{O}_Y)$ are invariants analogous to the genus of a curve. In classical terminology, $\dim H^2(Y, \mathcal{O}_Y)$ is the *geometric genus* p_g, and $\dim H^1(Y, \mathcal{O}_Y)$ is the *irregularity* q. The *arithmetic genus* of Y is

patwo (7.7.4)
$$p_a = \dim H^2(Y, \mathcal{O}_Y) - \dim H^1(Y, \mathcal{O}_Y)$$

If Y is a surface in $\mathbb{P}^3$, then $p_g = p_a$ and $q = 0$, because $H^1(Y, \mathcal{O}_Y) = 0$.

support-dimzero

(7.7.5) $\mathcal{O}$-modules with support of dimension zero

Let X be a variety. We describe finite $\mathcal{O}$-modules whose supports are finite sets.

Say that the support of the finite $\mathcal{O}$-module $\mathcal{M}$ is the set $\{p_1, ..., p_k\}$. We choose affine open subset $U^i = \operatorname{Spec} A_i$ of X such that p_i is a point of U^i, but p_ν is not in U^ν if $j \neq i$. Then we choose other affine opens $U^{k+1}, ..., U^n$ that con't contain any of the points p_i, so as to obtain a covering $U^1, ..., U^n$ of X. Let M_i denote the finite A_i-module $\mathcal{M}(U^i)$, and let $M_{ij} = \mathcal{M}(U^i \cap U^j)$.

The A_i-module M_i has empty support when $i > k$. A module whose support is empty is zero, so $M_i = 0$ when $i > k$. The support of M_i is the point p_i if $i \leq k$. The zero locus of the annihilator I of M_i will be be p_i, so $\operatorname{rad} I = \mathfrak{m}_p$, and $\mathfrak{m}_p^r \subset I$ if $r \gg 0$. This allows us to make M_i into a finite module over the ring $A_i/\mathfrak{m}_p^r$, and since this ring has finite dimension over $\mathbb{C}$, M_i will also have finite dimension.

Next, the support of $\mathcal{M}$ on $U^i \cap U^j$ will be empty if $i \neq j$, so $M_{ij} = 0$ if $i \neq j$. And since $U^i \cap U^i = U^i$, $M_{ii} = M_i$.

We write the exact sequence (6.2.17) that expresses the sheaf property for this covering. Taking into account the vanishing of many of the terms, it is

$$0 \to \mathcal{M}(X) \xrightarrow{\alpha} \prod_{i=1}^{k} M_i \xrightarrow{\beta} \prod_{i=1}^{k} M_i$$

and β is the zero map. Therefore $\mathcal{M}(X) = \prod M_i$.

This formula shows that the module M_i is independent of the choice of the affine open set U^i that contains the point p_i and none of the other points. Speaking loosely, one might say that $\mathcal{M}$ "is" the product $\prod M_i$ of the modules M_i.

We define the *multiplicity* of the module $\mathcal{M}$ at p_i to be the dimension of M_i as a vector space.

notextzero **7.7.6. Note.** When the support of a finite $\mathcal{O}$-module $\mathcal{M}$ is a single point p, it is tempting to think of $\mathcal{M}$ as the extension by zero of the $\mathcal{O}_p$-module M described above. This isn't accurate, because M has a structure of an $\mathcal{O}_X$-module that is different, unless the annihilator is equal to the maximal ideal $\mathfrak{m}_p$. □

bezout

7.8 Bézout's Theorem

We now prove a theorem that was stated in Chapter **??**. We repeat the statement here for reference, allowing reducible divisors:

bezoutrestated **7.8.1. B'ezout's Theorem.** *Let Y and Z be the divisors in the projective plane X defined by homogeneous polynomials f and g of degrees m and n with no common factor. The number of intersection points $Y \cap Z$, counted with appropriate multiplicity, is equal to mn.*

The definition of the multiplicity will emerge during the proof.

intersectlines **7.8.2. Example.** Suppose that f and g are products of distinct linear polynomials, so that Y is the union of m lines and Z is the union of n lines. In this case it is obvious that there are mn intersection points, and since distinct lines intersect transversally, the multiplicities are equal to 1. □

proof of Bézout's Theorem. To avoid clutter, we suppress notation for the extension by zero from a closed subset. Let $\mathcal{O}$ denote the structure sheaf $\mathcal{O}_X$. We begin by contructing a diagram of coherent sheaves. As we have seen (7.7.1) multiplication by f defines a short exact sequence of $\mathcal{O}_X$-modules

$$0 \to \mathcal{O}_X(-m) \xrightarrow{f} \mathcal{O}_X \to \mathcal{O}_Y \to 0$$

This sequence describes $\mathcal{O}(-m)$ as the ideal $\mathcal{I}$ of Y, and there is a similar sequence describing the ideal $\mathcal{J}$ of Z.

The intersection $Y \cap Z$ is the locus in X of the ideal $\mathcal{I} + \mathcal{J}$. Since f and g have no common factor, $Y \cap Z$ is a finite set. As a scheme, its coordinate algebra is $\mathcal{O}_X/(\mathcal{I} + \mathcal{J})$. We denote that algebra by $\mathcal{O}_{Y\cap Z}$. Its support is the finite set, say $\{p_1, ..., p_k\}$ of points making up the set $Y \cap Z$.

We define the multiplicity μ_i of $\mathcal{O}_{Y\cap Z}$ at p_i as in (**7.7.5**). Then the dimension of $H^0(\mathbb{P}^2, \mathcal{O}_{Y\cap Z})$ is the sum $\mu_1 + \cdots + \mu_k$, and $H^q(\mathbb{P}^2, \mathcal{O}_{Y\cap Z}) = 0$ for $q > 0$ (Theorem 7.6.5). We'll show that

$$\mu_1 + \cdots + \mu_k \;=\; \dim H^0(\mathbb{P}^2, \mathcal{O}_{Y\cap Z}) \;=\; \chi(\mathcal{O}_{Y\cap Z}) \;=\; mn.$$

This will prove Bézout's Theorem.

We form a sequence

bezoutres-olution (7.8.3)
$$0 \to \mathcal{O}_X(-m-n) \xrightarrow{(g,f)^t} \mathcal{O}_X(-m) \oplus \mathcal{O}_X(-n) \xrightarrow{(f,-g)} \mathcal{O}_X \xrightarrow{\pi} \mathcal{O}_{Y\cap Z} \to 0$$

in which the maps are self-explanatory.

bresolexact **7.8.4. Lemma.** *This sequence (7.8.3) is exact.*

proof. The sequence is a complex because $(f, -g)(g, f)^t = 0$, and the only place at which exactness isn't obvious is at $\mathcal{O}_X(-m) \oplus \mathcal{O}_X(-n)$. With the notation we have chosen for the maps, the sections of this module are column vectors. Say that a section $(u, v)^t$ over an open set U is in the kernel of the map $(f, -g)$: $fu = gv$. We write u as a homogeneous fraction p/q of degree $-m$, where q doesn't vanish at any point of U. Then $fp = gvq$ is an equation among homogeneous polynomials. Since f and g have no common factor, g divides p, say $p = gs$. Then $fs = vq$. The homogeneous fraction $w = s/q$ has degree $-m - n$. It is a section of $\mathcal{O}_X(-m - n)$ on U, and $(g, f)^t w = (u, v)^t$. □

Proposition 7.6.7, applied to the exact sequence (7.8.3), tells us that

chiforbezout (7.8.5) $$\chi(\mathcal{O}(-m-n)) - \chi(\mathcal{O}(-n)) - \chi(\mathcal{O}(-m)) + \chi(\mathcal{O}) - \chi(\mathcal{O}_{Y\cap Z}) \;=\; 0$$

is zero. This impies that the term $\chi(\mathcal{O}_{Y\cap Z})$ depends only on the integers m and n. Since we know that the answer is mn in one case, it is mn in every case. This completes the proof.

If you are suspicious of this reasoning, you can use Theorem 7.5.4 to compute (7.8.5). It gives

$$\chi(\mathcal{O}_{Y\cap Z}) \;=\; \binom{n+m-1}{2}, - \binom{m-1}{2} - \binom{n-1}{2} + 1$$

The right side works out to be mn. □

birkgroth 7.9 The Birkhoff-Grothendieck Theorem

This theorem describes torsion-free finite modules on the projective line.

Let M be a module over a domain A. A *torsion element* m of M is an element that is annihilated by some nonzero element a of A: $am = 0$. A *torsion module* is a module consisting entirely of torsion elements. If am is nonzero whenever a and m are nonzero elements of A and M, respectively, then M is *torsion-free*.

A torsion-free finite module over the polynomial ring in one variable $\mathbb{C}[t]$ is free, because $\mathbb{C}[t]$ is a principal ideal domain.

Translating to varieties, a finite, torsion-free module on the affine line $\mathbb{A}^1 = \operatorname{Spec}\mathbb{C}[t]$ is free, isomorphic to $\mathcal{O}_U^r$ for some k.

These definitions are extended to $\mathcal{O}_X$-modules by applying them to affine open sets.

torsfree-locfree **7.9.1. Corollary.** *Let $X = \mathbb{P}^1$. If a finite $\mathcal{O}_X$-module $\mathcal{M}$ is torsion-free, then it is locally free.* □

Since $\mathbb{P}^1$ has dimension 1, a finite torsion module on $\mathbb{P}^1$ will be supported on a finite set.

BGtheorem **7.9.2. Birkhoff-Grothendieck Theorem.** A finite, torsion-free $\mathcal{O}$-module $\mathcal{M}$ on the projective line $\mathbb{P}^1$ is a direct sum of twisting sheaves: $\mathcal{M} \approx \bigoplus \mathcal{O}(n_i)$.

This is Grothendieck's proof.

mapstoMbounded **7.9.3. Lemma.** *Let $\mathcal{M}$ be a torsion-free $\mathcal{O}$-module.*
(i) *The integers r for which there exists a nonzero map $\mathcal{O}(r) \to \mathcal{M}$ are bounded.*
(ii) *The integers r such that $H^0(X, \mathcal{M}(-r)) \neq 0$ are bounded.*

proof. **(i)** Since $\mathcal{M}$ is torsion-free, any nonzero map $\mathcal{O} \to \mathcal{M}$ will be injective, and since $\mathcal{O}(r)$ is locally isomorphic to $\mathcal{O}$, a nonzero map $\mathcal{O}(r) \to \mathcal{M}$ will be injective too. Therefore the associated map $H^0(X, \mathcal{O}(r)) \to H^0(X, \mathcal{M})$ will be injective. Then $\dim H^0(X, \mathcal{M}) \geq \dim H^0(X, \mathcal{O}(r)) = r + 1$. So r is bounded.

(ii) A nonzero section s of $H^0(X, \mathcal{M}(-r)) \neq 0$ defines a nonzero map $\mathcal{O} \to \mathcal{M}(-r)$. Its twist by r will be a nonzero map $\mathcal{O}(r) \to \mathcal{M}$. □

proof of Theorem 7.9.2 We may assume that $\mathcal{M}$ isn't the zero module. Since twisting is compatible with direct sums, we may at any time replace $\mathcal{M}$ by a twist $\mathcal{M}(n)$. As we know, the twist $\mathcal{M}(n)$ will be generated by its sections if n is sufficiently large, so it will have nonzero global sections. Lemma 7.9.3 **(ii)** shows that when we replace $\mathcal{M}$ by a suitable twist, we will have $H^0(X, \mathcal{M}) \neq 0$ but $H^0(X, \mathcal{M}(-1)) = 0$. We assume that this has been done.

We use induction on the *rank* of $\mathcal{M}$ as a locally free module. We choose a nonzero global section s of $\mathcal{M}$ and consider the injective map $\mathcal{O} \to \mathcal{M}$ it defines. Let $\mathcal{W}$ be the cokernel of this map, so that we have a short exact sequence

cvcwsequence (7.9.4) $$0 \to \mathcal{O} \to \mathcal{M} \to \mathcal{W} \to 0.$$

sectionbasic **7.9.5. Lemma.** **(i)** $H^0(X, \mathcal{W}(-1)) = 0$.
(ii) *$\mathcal{W}$ is a torsion-free $\mathcal{O}$-module.*
(iii) *$\mathcal{W}$ is a direct sum $\bigoplus_{i=1}^{k} \mathcal{O}(-r_i)$ of twisting sheaves on $\mathbb{P}^1$, with $r_i \geq 0$.*

proof. **(i)** This follows from the cohomology sequence associated to the twisted sequence

$$0 \to \mathcal{O}(-1) \to \mathcal{M}(-1) \to \mathcal{W}(-1) \to 0$$

because $H^0(X, \mathcal{M}(-1)) = 0$ and by Theorem 7.5.4), $H^1(X, \mathcal{O}(-1)) = 0$.

(ii) This follows from **(i)**. A torsion module with nonzero torsion has nonzero global sections (see (**7.7.5**).

(iii) It follows by induction on the rank that $\mathcal{W}$ is a sum of twisting sheaves, and $r_i \geq 0$ because $H^0(X, \mathcal{W}(-1)) = 0$. □

Let n be the maximum of integers r_i, that appear in Lemma 7.9.5 **(iii)**. We twist again. Let $\mathcal{M}' = \mathcal{M}(n)$ and $\mathcal{W}' = \mathcal{W}(n)$. Then (7.9.4) gives us a short exact sequence

twistcvcw (7.9.6)
$$0 \to \mathcal{O}(n) \to \mathcal{M}' \to \mathcal{W}' \to 0,$$

Here $\mathcal{W}' = \bigoplus \mathcal{O}(s_i)$ with $s_i = n - r_i$. The integers s_i are non-negative, and at least one of them is zero. So $\mathcal{W}'$ has a summand isomorphic to $\mathcal{O}$. We project $\mathcal{W}'$ to one such summand, and form a diagram

$$\begin{array}{ccccc} \mathcal{O}(n) & \longrightarrow & \mathcal{M}' & \longrightarrow & \mathcal{W}' \\ & & \| & & \downarrow \\ & & \mathcal{M}' & \xrightarrow{\varphi} & \mathcal{O} \end{array}$$

where φ is the composed map. Since $H^1(X, \mathcal{O}(n)) = 0$, the map $H^0(X, \mathcal{M}') \to H^0(X, \mathcal{W}')$ is surjective, and so is the map $H^0(X, \mathcal{M}') \to H^0(X, \mathcal{O})$. So there is a section $m \in H^0(X, \mathcal{M}')$ whose image $\varphi(m)$ in $\mathcal{O}$ is is 1. This section defines a map $\mathcal{O} \to \mathcal{M}'$ whose composition with φ is the identity. It splits the map φ. It shows that $\mathcal{M}'$ is isomorphic to $\mathcal{O} \oplus \mathcal{N}$ for some $\mathcal{N}$. By induction on the rank, $\mathcal{N}$ is a sum of twists of $\mathcal{O}$, and so are $\mathcal{M}'$ and $\mathcal{M}$. □

7.10 Exercises.

exerchap-seven **Section 1.**

Section 2.

Section 3.

Section 4.

Section 5.

Section 6.

Section 7.

Section 8.

Section 9.

==

Let M be a module over a finite-type domain A, and let α be an element of A. Prove that for all but finitely many complex numbers c, scalar multiplication by $s = \alpha - c$ is an injective map $M \xrightarrow{s} M$.

$H^q(X, \mathcal{O}_X(-k))$ The verification of this makes a good exercise.

Prove the Birkhoff-Grothendieck Theorem by matrix manipulation, as outlined in (7.10.1).

Before giving the proof we sketch a computational proof that is similar to Birkhoff's proof. His proof uses the standard affine cover $U^0 = \operatorname{Spec} \mathbb{C}[t]$ and $U^1 = \operatorname{Spec} \mathbb{C}[t^{-1}]$, and the intersection U^{01} is the spectrum of the Laurent polynomial ring $\mathbb{C}[t, t^{-1}]$. Let's write A_0, A_1, and A_{01} for the rings $\mathbb{C}[t], \mathbb{C}[t^{-1}]$, and $\mathbb{C}[t, t^{-1}]$, respectively, and let M_0, M_1, and M_{01} denote the modules of sections of $\mathcal{M}$ over the corresponding open sets. We know that M_0 and M_1 are free modules, so M_{01} is a free A_{01}-module. Let $\mathbf{B}_0$ and $\mathbf{B}_1$ be bases for M_0 and M_1. Both sets will be bases of M_{01}, so they will be related by an invertible A_{01}-matrix R. We can change the basis $\mathbf{B}_0$ by an invertible A_0-matrix Q and the basis $\mathbf{B}_1$ by an invertible A_1-matrix P. If we do this, R gets replaced by

qinversmp (7.10.1)
$$R' = Q^{-1} R P$$

Here Q and P can be arbitrary elements of $GL_n(A_0)$ and $GL_n(A_1)$. The theorem can be proved by showing that for any element R of $GL_n(A_{01})$, there exist elements Q in $GL_n(A_0)$ and P in $GL_n(A_1)$ such that $Q^{-1} R P$ is diagonal.

A morphism $Y @ > u >> X$ of varieties is an *affine morphism* if for every affine open subset U of X, $u^{-1} U$ is an affine open subset of Y. Prove that if u is an affine morphism and $\mathcal{N}$ is an $\mathcal{O}_Y$-module, then $H^q(Y, \mathcal{N}) \approx H^q(X, u_* \mathcal{N})$.

Exercise. for chapter 6 Let X and Y be varieties with functions fields K, L respectively. Suppose that L is a finite extension of K, of degree $[L : K] = n$.

(i) Show that is an integral morphism $Y' @ > u' >> X'$, where Y' and X' are nonempty open subvarieties of Y and X, respectively, such that every fibre of u' consists of exactly n points.

(ii) Show that if $K = L$, there are open subsets Y' and X' as in **(i)** such that the morphism $Y' \to X'$ is an isomorphism.

If $D \approx E$, multiplication by f defines an isomorphism $\mathcal{O}_Y(D) \xrightarrow{f} \mathcal{O}_Y(E)$.

Do Euler characteristic of complex.

$H^q(X, f_*\mathcal{N}) = H^q(Y, \mathcal{N})$ if $Y \xrightarrow{f} X$ is an affine morphism.

Let $x_0, ..., x_n$ be coordinates in projective space $\mathbb{P}^n$, and let $\mathcal{M}$ be an $\mathcal{O}_{\mathbb{P}^n}$-module. Let M be the module of sections of $\mathcal{M}$ on the standard affine open $U^0 = \operatorname{Spec} \mathbb{C}[u_1, ..., u_n]$, $u_i = x_i/x_0$. When the coordinates are in general position, multiplication by u_i defines an injective map $M \to M$.

support

general position

dimension zero loci redundancy in BG theorem