

Chapter 6 MODULES

6.1 The Structure Sheaf

6.2 Modules

6.3 Defining a Module on a Basis

6.4 Homomorphisms

6.5 Direct Images

6.6 Twisting

6.7 Proofs

fnfld 6.1 The Structure Sheaf.

Regular functions on an open subset of a variety were defined in Section 3.5. As we saw there, the regular functions on an affine variety $X = \text{Spec } A$ are the elements of its algebra A , and the variety X is completely determined by that algebra. This isn't true for varieties that aren't affine, because they don't have enough regular functions. One has to look at the regular functions on affine open subsets. The structure sheaf of a variety X organizes things by keeping track, implicitly, of the regular functions on every open subset.

defstrsheaf **6.1.1. Definition.** The *structure sheaf* \mathcal{O}_X on a variety X is the map

defstrsheaftwo (6.1.2)
$$(\text{opens})^\circ \xrightarrow{\mathcal{O}_X} (\text{algebras})$$

from open sets to algebras that sends a nonempty open set U to the algebra, denoted by $\mathcal{O}_X(U)$, of rational functions that are regular on U . So $\mathcal{O}_X(U)$ is a subalgebra of the function field K of X .

Elements of $\mathcal{O}_X(U)$ are called *sections* of $\int \mathcal{O}_X$ on U , and elements of $\mathcal{O}_X(X)$, functions that are regular everywhere, are *global sections*.

We make the set (opens) of open subsets of X into a category, defining morphisms between open sets to be inclusion maps. So if $V \subset U$ there is a unique morphism $V \rightarrow U$, and if $V \not\subset U$ there is no such morphism.

The structure sheaf \mathcal{O}_X is a *sheaf* because it has the following two properties, both obvious:

strsheafprop (6.1.3)

- \mathcal{O}_X is a contravariant *functor*: If $V \subset U$ are nonempty open sets, then regular function on U is also regular on V : $\mathcal{O}_X(V) \supset \mathcal{O}_X(U)$.

(The superscript \circ in (6.1.2) indicates that arrows, inclusions here, are reversed by \mathcal{O}_X , i.e., that the functor is contravariant.)

- \mathcal{O}_X is a *sheaf*: If V_1, \dots, V_k are nonempty open subsets that cover another open set V , i.e., if $V = \bigcup V_i$, a rational function is regular on V if and only if it is regular on each V_i : $\mathcal{O}_X(V) = \bigcap \mathcal{O}_X(V_i)$.

This sheaf property will be discussed in more detail below.

pnminuspoint **6.1.4. Example.** Let Z be the locus of zeros of a homogeneous polynomial f of degree r on \mathbb{P}^n . Corollary 3.4.6 describes the sections of the structure sheaf on the complement U of Z . The sections on U are fractions h/f^k , where h is homogeneous, of degree rk . \square

Obviously, one doesn't want to compute the regular functions on every open set U , and one never does. As we will see, the structure sheaf is determined by the regular functions on a collection of affine open sets that cover X , such as on the standard affine covering of \mathbb{P}^n . Moreover, one rarely needs to determine the functions explicitly, even on this covering.

When studying the structure sheaf, one always works with affine open sets.

- *Information about an open set that isn't affine is determined by the sheaf property.*

module **6.2 Modules**

modulehoms **(6.2.1) module homomorphisms**

Because we will be working with modules over various rings, we define homomorphisms of modules whose rings of scalars may be different. Let M and M' be modules over rings R and R' , respectively. A *homomorphism* $M \xrightarrow{\varphi} M'$ consists of a homomorphism of additive groups $M \xrightarrow{\varphi} M'$ and a ring homomorphism $R \xrightarrow{\rho} R'$, the two maps being compatible, meaning that, for m in M and a in R ,

homwithrho (6.2.2)
$$\varphi(am) = \rho(a)\varphi(m)$$

This condition is expressed by the diagram below, in which the horizontal arrows represent scalar multiplication:

$$\begin{array}{ccc} R \times M & \longrightarrow & M \\ (\rho, \varphi) \downarrow & & \downarrow \varphi \\ R \times M' & \longrightarrow & M' \end{array}$$

For example, if M is a module over a domain R and s is a nonzero element of R , the localization M_s is an R_s -module, and there is a homomorphism $M \rightarrow M_s$ that sends m to m , the associated map $R \rightarrow R_s$ being the inclusion.

As in this example, the ring homomorphism involved in a module homomorphism is often obvious. If so, we may not mention it explicitly.

The more usual concept of a homomorphism $M \xrightarrow{\varphi} M'$ of modules over the same ring R is the case that ρ is the identity map: $\varphi(am) = a\varphi(m)$. For clarity, we may refer to such map as a *homomorphism of R -modules*.

The case that $M' \xrightarrow{\varphi} M'$ is the identity map, but $R \xrightarrow{\rho} R'$ is arbitrary: $am' = \rho(a)m'$ is called the *restriction of scalars* from R' to R in an R' -module M' .

The term "restriction of scalars" is adapted from the special case that R is a subring of R' .

A module homomorphism $M \xrightarrow{\varphi} M'$ associated to a ring homomorphism $R \xrightarrow{\rho} R'$ becomes a homomorphism of R -modules when M' is made into an R -module by restriction of scalars.

omod **(6.2.3) \mathcal{O} -modules**

As above, the structure sheaf $\mathcal{O} = \mathcal{O}_X$ of a variety X is the functor $(opens)^\circ \rightarrow (algebras)$ that associates to each nonempty open subset U of X the algebra $\mathcal{O}(U)$ of regular functions on U , and to each inclusion $V \xrightarrow{j} U$ of open sets the injective map $\mathcal{O}(U) \rightarrow \mathcal{O}(V)$.

An \mathcal{O} -module \mathcal{M} is a functor

functmod (6.2.4)
$$(opens)^\circ \xrightarrow{\mathcal{M}} (modules)$$

that associates to each open set U an $\mathcal{O}(U)$ -module $\mathcal{M}(U)$, and to each inclusion of open sets $V \xrightarrow{j} U$ a homomorphism of modules $\mathcal{M}(U) \xrightarrow{j^*} \mathcal{M}(V)$ in the opposite direction, the associated homomorphism

of algebras being the inclusion $\mathcal{O}(U) \subset \mathcal{O}(V)$. The homomorphism j^* is called *restriction*. It needn't be injective.

Elements of $\mathcal{M}(U)$ are called *sections* of \mathcal{M} on U , and elements of $\mathcal{M}(X)$ are *global sections*. If $V \rightarrow U$ is an inclusion of open sets and m is a section on U , its image j^*m in $\mathcal{M}(V)$ is the *restriction* of m to V .

The restriction of a section m may be denoted, as above, by j^*m . However, because the operation of restriction occurs very often, we usually abbreviate, using the same symbol m for a section and for its restriction. Also, if an open set V is contained in two other open sets U^1 and U^2 , and if m_i is a section of \mathcal{M} on U^i , we say that m_1 and m_2 are *equal on V* if their restrictions to V are equal.

A *homomorphism* $\mathcal{M} \rightarrow \mathcal{N}$ of \mathcal{O} -modules consists of a homomorphism of $\mathcal{O}(U)$ -modules $\mathcal{M}(U) \rightarrow \mathcal{N}(U)$ for each open set U . These homomorphisms are required to be compatible with restriction when $V \subset U$, as is indicated by the diagram

$$\begin{array}{ccc} \mathcal{M}(U) & \longrightarrow & \mathcal{N}(U) \\ j^* \downarrow & & \downarrow j^* \\ \mathcal{M}(V) & \longrightarrow & \mathcal{N}(V) \end{array}$$

The simplest examples of \mathcal{O} -modules are the *free* \mathcal{O} -module \mathcal{O}^n , whose sections on an open set U form the free $\mathcal{O}(U)$ -module $\mathcal{O}(U)^n$.

Multiplication by a global section α of \mathcal{O} , a regular function on X , defines a homomorphism $\mathcal{O} \rightarrow \mathcal{O}$: If U is an open set, α will be a regular function on U , so multiplication by α is an $\mathcal{O}(U)$ -linear map $\mathcal{O}(U) \rightarrow \mathcal{O}(U)$. Thus a homomorphism $\mathcal{O}^m \rightarrow \mathcal{O}^n$ of free modules will be given by an $m \times n$ -matrix of global sections of \mathcal{O} .

defqcoh

6.2.5. Definition. An \mathcal{O} -module \mathcal{M} is *quasicoherent* if it satisfies the following two conditions:

the coherence property (*compatibility with localization*)

Let U be an open subset of X , and let $R = \mathcal{O}(U)$ be the algebra of regular functions on U . Let s be a nonzero element of R , and let U_s be the open subset of U points at which $s \neq 0$. If \mathcal{M} is an \mathcal{O} -module and $\mathcal{M}(U)$ is the R -module M , then $\mathcal{M}(U_s)$ is its localization M_s .

If this property is true for a particular open set U , we may say that the coherence property holds for U .

the sheaf property (*sections are determined locally*)

Let U be an open subset of X and let $\{U^i\}$ be open subsets that cover U . Let $m_i \in \mathcal{M}(U^i)$ be sections of \mathcal{M} on U^i . If $m_i = m_j$ on $U^i \cap U^j$ for all i and j , there is a unique section m of \mathcal{M} on U such that $m = m_i$ on U^i for all i .

This property is also called the *sheaf axiom*. A functor that has this property is a *sheaf*.

For the structure sheaf \mathcal{O} , the coherence property asserts that, with notation as above, $\mathcal{O}(U_s) = R_s$. This is true when U is an affine open set, and Theorem 6.3.1 below asserts that it is true for arbitrary open sets.

The sheaf property is trivial for the structure sheaf: If $\{U^i\}$ cover U and if a rational function α is regular on each U^i then α is regular on U .

coverpone

6.2.6. Example. The projective line \mathbb{P}^1 is covered by the two standard affine open sets \mathbb{U}^0 and \mathbb{U}^1 , and with our usual notation, $\mathcal{O}(\mathbb{U}^0) = \mathbb{C}[t]$, $\mathcal{O}(\mathbb{U}^1) = \mathbb{C}[u]$, and $\mathcal{O}(\mathbb{U}^0 \cap \mathbb{U}^1) = \mathbb{C}[t, u]$, where $u = t^{-1}$. The sheaf property asserts that a global section of \mathcal{O} is determined by elements $a(t) \in \mathbb{C}[t]$ and $b(u) \in \mathbb{C}[u]$ such that $a(t) = b(u)$ in $\mathbb{C}[t, u]$. Since $u = t^{-1}$, a and b must be equal constants. The only rational functions that are regular everywhere on \mathbb{P}^1 are the constants. I think we knew this. \square

The coherence property is more fundamental than the sheaf property. As Theorem 6.3.6 below shows, the sheaf property follows from the coherence property on an affine variety X . The sheaf property is needed only to determine the sections of an \mathcal{O} -module on the subsets that aren't affine.

When working with \mathcal{O} -modules, one should **always** work with affine open sets. Though we will be interested in the global sections of a module, the non-affine opens are just along for the ride most of the time.

We make a few more preliminary remarks here.

As the example above shows, a module may have few sections on a non-affine open set. The global sections of $\mathcal{O}_{\mathbb{P}^1}$ are the constants, and they tell us very little about the structure sheaf. On the other hand, the regular functions on the affine line \mathbb{A}^1 form the polynomial ring $\mathbb{C}[t]$, and that ring determines the structure of \mathbb{A}^1 completely.

In the coherence property, saying $\mathcal{M}(U_s)$ “is” the localization M_s is inaccurate. It would be better to say that M_s and $\mathcal{M}(U_s)$ are canonically isomorphic. When \mathcal{M} is a functor, the inclusion of $U_s \subset U$ gives us a module homomorphism $\mathcal{M}(U) = M \rightarrow \mathcal{M}(U_s)$, and since s is invertible on U_s , this map extends to a homomorphism $M_s \rightarrow \mathcal{M}(U_s)$. The coherence condition requires that this homomorphism be an isomorphism. To avoid cluttering up our notation, we often identify canonically isomorphic objects.

emptyset

6.2.7. Proposition. *The only section of an \mathcal{O}_X -module \mathcal{M} on the empty set is the zero section: $\mathcal{M}(\emptyset) = 0$. In particular, $\mathcal{O}_X(\emptyset) = 0$.*

proof. This follows from the sheaf property. The empty set \emptyset is covered by the empty covering, the family indexed by the empty set. The only set of sections on the empty covering is the empty set. The sheaf property asserts that this empty set of sections determines a unique section on \emptyset . So $\mathcal{M}(\emptyset)$ contains a unique element. That element is zero. \square

sheafonpoint

6.2.8. Example. The affine variety $\text{Spec } \mathbb{C}$ is a point; let’s call it p . It has two open sets: the whole space p and the empty set, and since $\mathcal{O}(p) = \mathbb{C}$, the global sections $\mathcal{M}(p)$ of an \mathcal{O} -module \mathcal{M} form a complex vector space, while $\mathcal{M}(\emptyset) = 0$. The sheaf and the coherence properties are trivial, so to define \mathcal{M} , one can assign the vector space $\mathcal{M}(p)$ arbitrarily. Speaking informally, we may say that a module on the point *is* a complex vector space. \square

somemodules

(6.2.9) the twisting modules

The twisting modules are among the most important modules on projective space.

Let X be the projective space with coordinates x_0, \dots, x_n . If g and h are homogeneous polynomials in x of the same degree, the fraction g/h defines a rational function on X . The nonzero elements of the function field of X are the equivalence classes of such fractions (3.4.5). To define the twisting modules $\mathcal{O}(d)$, we consider *homogeneous fractions*, fractions of homogeneous polynomials that aren’t necessarily of the same degree. The *degree* of a nonzero homogeneous fraction g/h is the difference $\deg g - \deg h$. It can be any integer. The set of homogeneous fractions of degree d , together with the zero element, will be denoted by \mathcal{K}_d . So \mathcal{K}_0 is the function field of X .

An element of \mathcal{K}_d is *regular* at a point p if it can be represented as a fraction g/h whose denominator h isn’t zero at p , and an element is regular on an open set U if it is regular at every point of U . When we write a fraction in the form g/h where g and h are homogeneous and relatively prime, it will be regular at p if and only if $h(p) \neq 0$. This agrees with the definition given before for rational functions. For example, the only homogeneous polynomials that don’t vanish at any point of the standard affine \mathbb{U}^0 are the powers of x_0 . So the homogeneous fractions that are regular on \mathbb{U}^0 have the form g/x_0^k .

The twisting module $\mathcal{O}(d)$ on \mathbb{P}^n is defined as follows:

The sections of $\mathcal{O}(d)$ on an open set U are the homogeneous fractions of degree d that are regular on U .

Odismodule

6.2.10. Lemma. (i) *The twisting module $\mathcal{O}(d)$ on projective space X is a quasicoherent \mathcal{O} -module.*
(ii) *The sections of $\mathcal{O}(d)$ on the standard affine open set \mathbb{U}^0 form a free module of rank one over the polynomial ring $\mathcal{O}(\mathbb{U}^0)$, with basis x_0^d .*

Thus $\mathcal{O}(d)$ seems quite similar to the structure sheaf. However, $\mathcal{O}(d)$ is only *locally* free. The sections of $\mathcal{O}(d)$ on the standard open set \mathbb{U}^1 form a free $\mathcal{O}(\mathbb{U}^1)$ -module with basis x_1^d . The bases on the two open sets are related by the factor $(x_0/x_1)^d$.

(A property \mathcal{P} is said to be true “locally” on a topological space X if there is an open covering $\{U^i\}$ of X such that \mathcal{P} is true on every U^i .)

proof of Lemma 6.2.10 (i) It is easy to show that $\mathcal{O}(d)$ is a functor and that the sheaf property holds. We verify the coherence property. Let R be the coordinate ring of an open subset U of X , and let s be a nonzero element of R . We must show that the module of sections of $\mathcal{O}(d)$ on U_s is obtained by localizing the module of sections on U . The essential point to show is that, if u is a section of $\mathcal{O}(d)$ on U_s , then for sufficiently large k , $s^k u$ is a section on U .

Say that we write $s = q/r$ and $u = g/h$ as homogeneous fractions, in which r doesn't vanish on U and h doesn't vanish on U_s . Then $s^k u$ is regular on U if q^k/h is regular on U . So it is enough to show that, if a homogeneous polynomial h is regular on U_s , then for large k , $q^k h$ is regular on U .

We may assume that h is an irreducible polynomial. Let Y be the closed complement of U in X , and let $Z = V_X(q)$ be the zero locus of q . The union $Y \cup Z$ is the complement of U_s in X , and it contains the irreducible closed set $C = V_X(h)$. Therefore C is contained in Y or in Z . If $C \subset Y$, then $1/h$ is regular on U_s . If $V_X(h) \subset Z = V_X(q)$, then by the Strong Nullstellensatz, h divides a power q^k of q , and q^k/h is regular on U . \square

6.2.11. Proposition. *When $d \geq 0$, the global sections of the twisting module $\mathcal{O}(d)$ on \mathbb{P}^n , $n > 0$, are the homogeneous polynomials of degree d . When $d < 0$, the only global section of $\mathcal{O}(d)$ is zero.*

proof. A global section u of $\mathcal{O}(d)$ will restrict to a section on the standard affine open set \mathbb{U}^0 , which means that u is a homogeneous fraction whose denominator is a power of x_0 . Similarly, restriction to \mathbb{U}^1 shows that u is a homogeneous fraction whose denominator is a power of x_1 . So u can have no denominator. \square

As we see here, it is often convenient to describe a quasicohherent \mathcal{O} -module \mathcal{M} by its sections on an affine open covering $\{U^i\}$, such as on the standard covering of projective space. When $\mathcal{M}(U^i)$ is known, the coherence property determines the sections on localizations U_s^i , and the sections on other open sets are determined by the sheaf property. This is the *standard procedure* to describe a module.

The product uv of homogeneous fractions of degrees r and s is a homogeneous fraction of degree $r + s$, and if u and v are regular on an open set U , so is uv . Therefore multiplication defines a homomorphism of \mathcal{O} -modules

$$(6.2.12) \quad \mathcal{O}(r) \times \mathcal{O}(s) \rightarrow \mathcal{O}(r + s)$$

Multiplication by a homogeneous polynomial f of degree d defines an injective homomorphism

$$(6.2.13) \quad \mathcal{O}(k) \xrightarrow{f} \mathcal{O}(k + d).$$

When we set $k = -d$, we obtain a homomorphism $\mathcal{O}(-d) \xrightarrow{f} \mathcal{O}$. Or, setting $k = 0$ gives us a homomorphism $\mathcal{O} \xrightarrow{f} \mathcal{O}(d)$.

(6.2.14) ideals

An *ideal* \mathcal{I} in the structure sheaf of a variety X is a quasicohherent submodule of $\mathcal{O} = \mathcal{O}_X$. For example the structure sheaf has a *maximal ideal* at a point p . We'll denote this ideal by \mathfrak{m}_p , using the same notation as for maximal ideals of algebras. The sections of \mathfrak{m}_p on an open set U are as follows: If U contains p , then $\mathfrak{m}_p(U)$ is the ideal of $\mathcal{O}(U)$ of regular functions that vanish at p , while if U doesn't contain p , $\mathfrak{m}_p(U)$ is the unit ideal $\mathcal{O}(U)$.

The main example is the ideal of functions that vanish on a closed subset Y of X . Its sections on an open set U are the regular functions on U that are zero at every point of $Y \cap U$. Let \mathcal{I} denote this ideal. The sheaf property for \mathcal{I} is fairly obvious. Let's assume that the coherence property has been shown for the structure sheaf \mathcal{O} (see Theorem 6.3.1 below), and deduce it for \mathcal{I} .

Let s be a nonzero element of $R = \mathcal{O}(U)$. The coherence property for \mathcal{O} asserts that $\mathcal{O}(U_s) = R_s$. Let $I = \mathcal{I}(U)$ be the ideal of R of regular functions that vanish on $Y \cap U$. We must show that $\mathcal{I}(U_s)$ is the localization I_s . Because $I \subset \mathcal{I}(U_s)$ and s is invertible in R_s , $I_s \subset \mathcal{I}(U_s)$. Let α be an element of $\mathcal{I}(U_s)$. So α is an element of R_s that is zero on $Y \cap U_s$. Since s is invertible in R_s , we may clear the denominator in α to reduce to the case that α is in R . Let Z be the zero locus of s in U . Since α is zero on $Y \cap U_s$ and s is zero on Z , αs is zero on $Y \cap U$. So αs is in I , and α is in I_s . \square

We'll describe the ideals in the structure sheaf of the projective plane $X = \mathbb{P}^2$ that correspond to plane curves more completely. We restrict to the plane only to focus attention. An analogous discussion can be made for the ideals of hypersurfaces in projective spaces of arbitrary dimension.

curveideal

6.2.15. Lemma. *Let C be the plane curve defined by an irreducible homogeneous polynomial f of degree d , and let \mathcal{I} be the ideal of C .*

(i) *The sections of \mathcal{I} on an open set U are the rational functions that can be written as fractions g/h of homogeneous polynomials, such that f divides g and h doesn't vanish at any point of U .*

(ii) *Let $\mathcal{O}(-d)$ be the twisting module. The image of the multiplication map $\mathcal{O}(-d) \xrightarrow{f} \mathcal{O}$ is the ideal \mathcal{I} . Thus, as \mathcal{O} -module, \mathcal{I} is isomorphic to the twisting sheaf $\mathcal{O}(-d)$. The ideals of all plane curves of degree d are isomorphic \mathcal{O} -modules.*

proof. We represent a rational function u as a fraction g/h of relatively prime homogeneous polynomials. To be a section of \mathcal{I} on U , u must satisfy two conditions: First, u must be a regular function on U . So the denominator h cannot vanish at any point of U . Suppose that this is so. Then there are two cases. If the intersection $C' = C \cap U$ isn't empty, it will be a dense open subset of the irreducible closed set C . The numerator g must vanish at every point of C' , and since C' is dense, it will vanish on C . This is true if and only if f divides g . If C' is empty, any regular function u on U will be a section of \mathcal{I} . In that case f won't vanish at any point of U , and therefore g/h can also be written as the fraction fg/fh whose numerator is divisible by f . This proves (i), and (ii) follows easily. \square

basicmod-prop

(6.2.16) the sheaf property as an exact sequence

Let U be an open subset of a variety X and U^i be open subsets that cover U . The sheaf property for an \mathcal{O}_X -module and for this covering can be expressed as an exact sequence

sheafzero

$$(6.2.17) \quad 0 \rightarrow \mathcal{M}(U) \xrightarrow{\alpha} \prod_i \mathcal{M}(U^i) \xrightarrow{\beta} \prod_{i,j} \mathcal{M}(U^i \cap U^j)$$

Let

in which the maps α and β are explained as follows. To simplify notation, let's suppose that there are finitely many sets forming the covering, say U^1, \dots, U^k . Then elements of $\prod \mathcal{M}(U^i)$ can be viewed as k -dimensional vectors, and elements of $\prod \mathcal{M}(U^i \cap U^j)$ can be viewed as $k \times k$ matrices.

definition of α : Let s be a section of $\mathcal{M}(U)$. The i th coordinate of the vector $\alpha(s)$ is the restriction of s to U^i .

definition of β : Let $m = (m_1, \dots, m_k)$ be an element of $\prod \mathcal{M}(U^i)$. The i, j entry z_{ij} of the matrix $\beta(m)$ is the difference of the restrictions of m_j and m_i to $U^i \cap U^j$. Using our abbreviated notation for restricted sections, $z_{ij} = m_j - m_i$ on $\mathcal{M}(U^i \cap U^j)$.

The sheaf property asserts that the map α is injective, and that its image is the kernel of β . Thus sections of \mathcal{M} over U correspond bijectively to vectors (m_1, \dots, m_k) with $m_i \in \mathcal{M}(U^i)$, such that $m_i = m_j$ on $U^i \cap U^j$.

ilessthanj

6.2.18. Note. A vector (m_1, \dots, m_k) of sections of \mathcal{M} is in the kernel of β if the property $m_i = m_j$ on $U^i \cap U^j$ is true for the pairs of indices i, j with $i < j$. The index pairs with $i \geq j$ are redundant because the subsets $U^j \cap U^i$ and $U^i \cap U^j$ are the same, and it is trivial that $m_i = m_i$ on $U^i \cap U^i$. \square

twoopens

6.2.19. Corollary. *Let \mathcal{M} be a quasicoherent \mathcal{O} -module and let U be an open set that is covered by two open sets U^0 and U^1 . A pair of sections m_0, m_1 on the two open sets determines a section on U if $m_0 = m_1$ on $U^0 \cap U^1$, or if the sequence*

$$0 \rightarrow \mathcal{M}(U) \rightarrow \mathcal{M}(U^0) \oplus \mathcal{M}(U^1) \xrightarrow{\begin{smallmatrix} \tau \\ \tau \end{smallmatrix}} \mathcal{M}(U^0 \cap U^1)$$

is exact. \square

module-
one

6.2.20. Example. We go back to the covering of the projective line \mathbb{P}^1 by the standard affine open sets $\mathbb{U}^0 = \text{Spec } \mathbb{C}[t]$ and $\mathbb{U}^1 = \text{Spec } \mathbb{C}[t^{-1}]$. Their intersection \mathbb{U}^{01} is the spectrum of the *Laurent polynomial ring* $\mathbb{C}[t, t^{-1}]$, whose elements are combinations of powers of t , negative powers included. Let \mathcal{M} be a quasicohherent \mathcal{O} -module, and let $M_i = \mathcal{M}(\mathbb{U}^i)$. The coherence property of \mathcal{M} tells us that M_{01} can be obtained by localization from M_0 and also from M_1 :

$$M_0[t^{-1}] \approx M_{01} \approx M_1[t].$$

A global section of \mathcal{M} is determined by a pair of elements $m_0 \in M_0$ and $m_1 \in M_1$ that become equal in this common localization. □

basis

6.3 Defining a Module on a Basis

The theorems in this section show that, in order to determine a quasicohherent \mathcal{O} -module \mathcal{M} , one doesn't need to know the sections on every open set. It is enough to know the sections on a particular affine covering. The proofs are in Section 6.7.

onecovere-
nough

6.3.1. Theorem. *Let $\{U^i\}$ be an affine open covering of a variety X , and let \mathcal{M} be an \mathcal{O} -module. Suppose that \mathcal{M} has the sheaf property, and that the coherence property is true for the open sets U^i . Then the coherence property is true for every open subset, and therefore \mathcal{M} is quasicohherent.*

coherent

6.3.2. Corollary. *The structure sheaf \mathcal{O} has the coherence property.*

The corollary follows from the fact that the coherence property for \mathcal{O} is true for affine open sets. □

A collection \mathcal{B} of open subsets of a topological space X is a *basis* if every open subset of X can be covered by subsets that are in \mathcal{B} . We may consider a basis as a subcategory of the category of open sets, morphisms being inclusions.

defgb

6.3.3. Definition. A set \mathcal{B} of open subsets of a variety X is a *good basis* if it has these properties:

- There is a covering of X by affine open sets that are in \mathcal{B} .
- If U and V are in \mathcal{B} , then $U \cap V$ is in \mathcal{B} .
- If an affine open set U is in \mathcal{B} , then every localization U_s of U is in \mathcal{B} .

The localizations U_s of an affine variety X form a good basis. If $\{U^i\}$ is a covering of an arbitrary variety X by affine open sets, the affine open sets that are contained in one of the sets U^i form a good basis (see 3.5.11). In particular, the set of all affine open subsets of X is a good basis. These are the most important examples.

Let

$$(6.3.4) \quad \mathcal{B}^\circ \xrightarrow{\mathcal{M}} (\text{modules})$$

be a functor such that $\mathcal{M}(U)$ is an $\mathcal{O}(U)$ -module for every U in \mathcal{B} . We say that \mathcal{M} is a *quasicohherent \mathcal{B} -module* if the sheaf property (6.7.3) is true for every covering of an open set U by \mathcal{B} -open sets $\{U^i\}$, and if the coherence property holds for open sets in \mathcal{B} .

sheafonbasis

6.3.5. Theorem. *Let \mathcal{B} be a good basis for the topology on a variety X , and let \mathcal{M} be a quasicohherent \mathcal{B} -module. There is a quasicohherent \mathcal{O} -module, unique up to unique isomorphism, whose restriction to \mathcal{B} is \mathcal{M} .*

The analogue of this theorem is true for sheaves of abelian groups on any topological space and for any basis, but never mind.

The theorem is almost obvious. Suppose that \mathcal{M} is given on a good basis, and let U be any open set. We cover U by opens U^i that are in \mathcal{B} , and then the intersections $U^i \cap U^j$ will be in \mathcal{B} too. The sheaf property for the \mathcal{O} -module that we hope to define requires that the sequence (6.2.17)

$$0 \rightarrow \mathcal{M}(U) \rightarrow \prod \mathcal{M}(U^i) \rightarrow \prod \mathcal{M}(U^{ij})$$

be exact. This identifies $\mathcal{M}(U)$ as the kernel of a map between modules that are given. But we must verify that this defines $\mathcal{M}(U)$ up to unique isomorphism, that it gives a functor, and that this functor has the sheaf property. These points are explained in the last section of the chapter. Then the coherence property follows from Theorem 6.3.1.

The next theorem and its corollary show that the sheaf property for an affine variety is a consequence of the coherence property.

6.3.6. Theorem. *Let s_1, \dots, s_k be elements of a ring A that generate the unit ideal. Let M be an A -module, and let M_i and M_{ij} denote the localizations M_{s_i} and $M_{s_i s_j}$, respectively. The sequence*

$$0 \rightarrow M \xrightarrow{\alpha} \prod M_i \xrightarrow{\beta} \prod M_{ij}$$

analogous to (6.2.17) is exact.

The relation of this theorem with \mathcal{O} -modules is as follows: Say that $U = \text{Spec } A$ and let $U^i = U_{s_i}$. Let \mathcal{M} be an \mathcal{O} -module that has the coherence property for U , and let $M = \mathcal{M}(U)$. Then $\mathcal{M}(U^i) = M_i$ and $\mathcal{M}(U^i \cap U^j) = M_{ij}$. The theorem asserts that the sequence (6.2.17) is exact, as is required by the sheaf property.

The next corollary tells us that the complication caused by working with \mathcal{O} -modules is justified only for varieties that aren't affine, because a quasicoherent \mathcal{O} -module on an affine variety is determined completely by its global sections.

6.3.7. Corollary. *On an affine variety $X = \text{Spec } A$, quasicoherent \mathcal{O} -modules and A -modules are equivalent concepts. \square*

Thus, to define a quasicoherent \mathcal{O} -module \mathcal{M} , it suffices to give its sections on the affine open subsets, and to verify the coherence property for those subsets. Since the affine open sets form a good basis, Theorem 6.3.5 tells us that the module is determined. This is the *standard procedure* for defining an \mathcal{O} -module. In fact, if $\{U^i\}$ is an affine open covering of X , the modules $M_i = \mathcal{M}(U^i)$ determine \mathcal{M} . However, one can't assign those modules arbitrarily because the open subsets U^i intersect. If V is an affine open set that is a localization of U^i and of U^j , say $V = U_s^i$ and $V = U_{s'}^j$, the localized modules $(M_i)_s$ and $(M_j)_{s'}$ are isomorphic.

6.3.8. Example. (modules on the projective line) We go back to the notation of Example 6.2.20. The two standard affine open sets $\mathbb{U}^0, \mathbb{U}^1$ and their intersection \mathbb{U}^{01} are spectra of the rings $A_0 = \mathbb{C}[t], A_1 = \mathbb{C}[t^{-1}]$, and $A_{01} = \mathbb{C}[t, t^{-1}]$. Let \mathcal{M} be a quasicoherent \mathcal{O} -module on \mathbb{P}^1 , and let M_0 and M_1 and M_{01} be the modules of sections on these open sets. The coherence property gives us isomorphisms

$$M_0[t^{-1}] \approx M_{01} \approx M_1[t]$$

The open subsets of \mathbb{P}^1 that are contained in \mathbb{U}^0 or in \mathbb{U}^1 are affine, and they form a good basis \mathcal{B} . If M_0 and M_1 are modules over A_0 and A_1 , respectively, and if an isomorphism $M_0[t^{-1}] \approx M_1[t]$ is given, one can use the coherence property to define an \mathcal{O} -module \mathcal{M} .

For instance, suppose that M_0 and M_1 are free modules of rank r over A_0 and A_1 . Then M_{01} will be a free A_{01} -module of rank r . A basis \mathbf{B}_0 of M_0 is also a basis of the A_{01} -module M_{01} , and a basis \mathbf{B}_1 of M_1 is also a basis of M_{01} . As bases of M_{01} , \mathbf{B}_0 and \mathbf{B}_1 will be related by an invertible A_{01} -matrix P , and this matrix determines the module up to isomorphism.

When $r = 1$, P will be an invertible 1×1 matrix in the Laurent polynomial ring $A_{01} = \mathbb{C}[t, t^{-1}]$, a unit of that ring. The units are scalar multiples of powers of t , and the scalar can be absorbed into one of the bases. Therefore quasicoherent \mathcal{O} -modules of rank 1 are determined, up to isomorphism, by powers of t . They are the twisting modules on \mathbb{P}^1 . \square

6.3.9. Terminology. There are \mathcal{O} -modules that aren't quasicoherent, but they are much less important, and we won't use them. To simplify terminology, and because the adjective "quasicoherent" is ugly, we will, from now on, refer to a quasicoherent \mathcal{O} -module simply as an *\mathcal{O} -module*. A *finite \mathcal{O} -module* \mathcal{M} is one such that $\mathcal{M}(U)$ is a finite $\mathcal{O}(U)$ -module for every *affine* open subset U of X . Elsewhere, finite \mathcal{O} -modules are called coherent sheaves. \square

finiteoncover

6.3.10. Proposition. *Let $\{U^i\}$ be an affine open covering of a variety X , let \mathcal{M} be an \mathcal{O} -module, and let $A_i = \mathcal{O}(U^i)$. If $M_i = \mathcal{M}(U^i)$ is a finite A_i -module for every i , then \mathcal{M} is a finite \mathcal{O} -module.*

The proof is in Section 6.7.

exactseq

6.4 Homomorphisms

As above, a homomorphism $\mathcal{M} \xrightarrow{\varphi} \mathcal{N}$ of \mathcal{O} -modules consists of homomorphisms of $\mathcal{O}(U)$ -modules

$$\mathcal{M}(U) \xrightarrow{\varphi(U)} \mathcal{N}(U)$$

for every open set U , that are compatible with restriction: If $V \xrightarrow{j} U$ is an inclusion of open sets, the diagram below commutes:

hommod

(6.4.1)

$$\begin{array}{ccc}
\mathcal{M}(U) & \xrightarrow{\varphi(U)} & \mathcal{N}(U) \\
j^* \downarrow & & j^* \downarrow \\
\mathcal{M}(V) & \xrightarrow{\varphi(V)} & \mathcal{N}(V)
\end{array}$$

For example, if m is a global section of an \mathcal{O} -module \mathcal{M} , the map $\mathcal{O} \xrightarrow{m} \mathcal{M}$ that sends a regular function α to αm is an example of a homomorphism. In particular, the map $\mathcal{O} \xrightarrow{f} \mathcal{O}(d)$ of multiplication by a homogeneous polynomial f of degree d is a homomorphism.

A sequence of homomorphisms

exseqsheaves

(6.4.2)

$$\mathcal{M} \rightarrow \mathcal{N} \rightarrow \mathcal{P}$$

of \mathcal{O} -modules on a variety X is *exact* if the sequence of sections

exactsections

(6.4.3)

$$\mathcal{M}(U) \rightarrow \mathcal{N}(U) \rightarrow \mathcal{P}(U)$$

is exact for every *affine* open subset U of X . This sequence isn't required to be exact for open subsets that aren't affine. However, the next theorem shows that the section functor $\mathcal{M} \rightsquigarrow \mathcal{M}(U)$ is always *left exact*. The proof will be given in Section 6.7.

leftexact-section

6.4.4. Theorem. *Let X be a variety, and let*

$$0 \rightarrow \mathcal{M} \rightarrow \mathcal{N} \rightarrow \mathcal{P}$$

be an exact sequence of \mathcal{O} -modules. For every open subset U of X , the sequence of sections

$$0 \rightarrow \mathcal{M}(U) \rightarrow \mathcal{N}(U) \rightarrow \mathcal{P}(U)$$

is exact.

When

$$0 \rightarrow \mathcal{M} \rightarrow \mathcal{N} \rightarrow \mathcal{P} \rightarrow 0$$

is a short exact sequence of \mathcal{O} -modules, the sequence of global sections

$$0 \rightarrow \mathcal{M}(X) \rightarrow \mathcal{N}(X) \rightarrow \mathcal{P}(X)$$

is often not exact when a zero is added on the right. Cohomology, the tool for analyzing the failure of exactness, will be discussed in Chapter 7.

cokerandim-
age

6.4.5. Example. Let X be the projective line with coordinates x_0, x_1 , and consider the sequence of homomorphisms

$$0 \rightarrow \mathcal{O}(-2) \xrightarrow{u} \mathcal{O}(-1) \oplus \mathcal{O}(-1) \xrightarrow{v} \mathcal{O} \rightarrow 0$$

where u sends a section s of $\mathcal{O}(-2)$ to $x_0s \oplus x_1s$ and v sends a section $a \oplus b$ of $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ to $x_1a - x_0b$. In matrix notation, $u = (x_0, x_1)^t$ and $v = (x_1, -x_0)$. You will be able to verify that this sequence is exact. Hence \mathcal{O} can be viewed as the cokernel of u , and also as the image of v . Since $\mathcal{O}(-1)$ has no nonzero global section, while $\mathcal{O}(X) = \mathbb{C}$, v does not define a surjective map of global sections. \square

The *kernel* \mathcal{K} , the *cokernel* \mathcal{C} , and the *image* \mathcal{I} of a homomorphism $\mathcal{M} \xrightarrow{\varphi} \mathcal{N}$ are defined by the standard procedure: looking on affine open sets. If U is an affine open set, then $\mathcal{K}(U)$, $\mathcal{C}(U)$, and $\mathcal{I}(U)$ are the kernel, cokernel, and image of the homomorphism $\mathcal{M}(U) \rightarrow \mathcal{N}(U)$. The coherence property carries over because localization is an exact operation, and because we have Theorem 6.3.1.

kernelsheaf

6.4.6. Corollary. Let \mathcal{K} be the kernel of a homomorphism of \mathcal{O} -modules $\mathcal{M} \rightarrow \mathcal{N}$. For every open set U , affine or not, $\mathcal{K}(U)$ is the kernel of the map $\mathcal{M}(U) \rightarrow \mathcal{N}(U)$. \square

The analogous statements for the cokernel and image are false.

A homomorphism of \mathcal{O} -modules $\mathcal{M} \xrightarrow{\varphi} \mathcal{N}$ is *injective* if its kernel is zero, which means that the map $\mathcal{M}(U) \xrightarrow{\varphi(U)} \mathcal{N}(U)$ is injective for every open set U . On the other hand, φ is *surjective* if its cokernel is zero, which means that $\varphi(U)$ is surjective for every *affine* open set U .

If $\{U^i\}$ is an affine open cover of a variety X , a homomorphism $\mathcal{M} \xrightarrow{\varphi} \mathcal{N}$ of \mathcal{O}_X -modules is surjective if the maps $\mathcal{M}(U^i) \rightarrow \mathcal{N}(U^i)$ are surjective for every i .

quotmod

(6.4.7) quotient modules

When \mathcal{M} is a submodule of an \mathcal{O} -module \mathcal{N} , the inclusion $\mathcal{M} \subset \mathcal{N}$ is a homomorphism. The *quotient module* \mathcal{N}/\mathcal{M} is the cokernel of the inclusion $\mathcal{M} \subset \mathcal{N}$. For example, if \mathfrak{m}_p is the maximal ideal of a point p of a variety X , then $\mathcal{O}/\mathfrak{m}_p$ is the residue field at p , which is made into an \mathcal{O} -module κ_p in an obvious way: If an open set U contains p , then $\kappa_p(U)$ is the residue field $k(p)$, and if U doesn't contain p , then $\kappa_p(U) = 0$.

directimage

6.5 Direct Images

Let $Y \xrightarrow{f} X$ be a morphism of varieties. The *direct image* $f_*\mathcal{N}$ of an \mathcal{O}_Y -module \mathcal{N} is the \mathcal{O}_X -module defined as follows: If U is an open subset of X , then

$$[f_*\mathcal{N}](U) \stackrel{def}{=} \mathcal{N}(f^{-1}U).$$

The \mathcal{O}_X -module structure on $f_*\mathcal{N}$ is obtained by restricting scalars. We spell this out:

Let $V = f^{-1}U$, and let $A = \mathcal{O}_X(U)$, $B = \mathcal{O}_Y(V)$, and $N_B = \mathcal{N}(V)$. (The subscript B is a reminder that N is a B -module.) The restriction of f to a morphism $V \rightarrow U$ gives us a homomorphism $A \xrightarrow{\rho} B$, and we make N_B into an A -module N_A by restricting scalars from B to A . Then $[f_*\mathcal{N}](U) = N_A$. As abelian groups, N_B and N_A are the same. So if α is in A and n is in N_A , then

$$\alpha n \stackrel{def}{=} \rho(\alpha)n$$

fstarcoh

6.5.1. Lemma. Let $Y \xrightarrow{f} X$ be a morphism of varieties. The direct image $f_*\mathcal{N}$ of an \mathcal{O}_Y -module \mathcal{N} is an \mathcal{O}_X -module.

Thus the *direct image* is a functor

fstarfunctor

$$(6.5.2) \quad (\mathcal{O}_Y\text{-modules}) \xrightarrow{f_*} (\mathcal{O}_X\text{-modules}).$$

proof of Lemma 6.5.1. It is easy to show that f_* is a functor. We must check the sheaf and coherence properties.

the sheaf property: Let $\{U^i\}$ be a covering of an open set U . The inverse images $V^i = f^{-1}U^i$ cover $V = f^{-1}U$. The sequence $0 \rightarrow \mathcal{N}(V) \rightarrow \prod \mathcal{N}(V^i) \rightarrow \prod \mathcal{N}(V^i \cap V^j)$ described in (6.7.3) is exact, and it is the same sequence of abelian groups as $0 \rightarrow [f_*\mathcal{N}](U) \rightarrow \prod [f_*\mathcal{N}](U^i) \rightarrow \prod [f_*\mathcal{N}](U^i \cap U^j)$.

the coherence property: With notation as above, let s be a nonzero element of A . The coherence property asserts that $[f_*\mathcal{N}](U_s)$ is the localization $(N_A)_s$ of the A -module N_A . Let s' be the image of s in B . There are two cases: If $s' \neq 0$, then $V_{s'} = f^{-1}U_s$. Since \mathcal{N} is quasicoherent, $\mathcal{N}(V_{s'})$ is the localization $(N_B)_{s'}$, and when we restrict scalars in, we obtain the localization $(N_A)_s$. Thus $[f_*\mathcal{N}](U_s) = \mathcal{N}(V_s) = (N_B)_{s'} = (N_A)_s$, as required. If $s' = 0$, $V_{s'}$ is the empty set, and $N_{B,s'} = \mathcal{N}(V_{s'}) = 0$. In this case, s annihilates N_A , so $N_{A,s} = 0$ too.

extbyzero (6.5.3) extension by zero

Let $Y \xrightarrow{i} X$ be the inclusion of a closed subvariety into a variety X . The direct image $i_*\mathcal{N}$ of an \mathcal{O}_Y -module \mathcal{N} is called the *extension by zero* of \mathcal{N} . If U is an open subset of X , then since i is an inclusion map, $i^{-1}U = U \cap Y$. Therefore

$$[i_*\mathcal{N}](U) = \mathcal{N}(U \cap Y)$$

The term “extension by zero” refers to the fact that, if an open set U of X doesn’t meet Y , then $U \cap Y$ is the empty set, and the module of sections of $i_*\mathcal{N}$ on U is zero. I think of the extension by zero as an operation that is essentially the identity. The space is the only thing that changes.

cokernot-sheaf

6.5.4. Example. (*the residue field*) (i) Let p be a point of a variety X , with residue field $k(p)$ ($= \mathbb{C}$), and let i denote the inclusion of p into X . A module on the one-point space $p = \text{Spec } k(p)$ is simply a vector space, and the residue field is such a module. We denote it by $k(p)$ too. The extension by zero $i_*k(p)$ from the closed set p is the \mathcal{O} -module that was denoted by κ_p above (6.4.7). If an open subset U of X contains p , then $\kappa_p(U) = k(p)$, and if U doesn’t contain p , $\kappa_p(U) = 0$.

(ii) Let X be the projective line \mathbb{P}_x^1 . The maximal ideal \mathfrak{m}_p fits into an exact sequence of \mathcal{O} -modules

$$(6.5.5) \quad 0 \rightarrow \mathfrak{m}_p \rightarrow \mathcal{O} \xrightarrow{\pi_p} \kappa_p \rightarrow 0$$

The spaces of global sections of these \mathcal{O} -modules happen to form a short exact sequence: $\mathfrak{m}_p(X) = 0$, and $\mathcal{O}(X) = \kappa_p(X) = \mathbb{C}$.

(iii) Let p and p' be distinct points of X , and let κ_p and $\kappa_{p'}$ denote the extensions by zero of the residue fields at these points. The intersection $\mathfrak{m}_p \cap \mathfrak{m}_{p'}$ of the maximal ideals fits into an analogous exact sequence of \mathcal{O} -modules

$$(6.5.6) \quad 0 \rightarrow \mathfrak{m}_p \cap \mathfrak{m}_{p'} \rightarrow \mathcal{O} \xrightarrow{\pi} \kappa_p \oplus \kappa_{p'} \rightarrow 0.$$

To see that π is surjective, we cover X by the open sets $U = X - p'$ and $U' = X - p$, both affine lines. It suffices to show that the restriction of π to these affine open sets is surjective (??), and this is true. On U , $\pi = \pi_p$, and on U' , $\pi = \pi_{p'}$. The sequence of global sections of (6.5.6) is left exact:

$$0 \rightarrow 0 \rightarrow \mathbb{C} \rightarrow \mathbb{C} \oplus \mathbb{C}$$

It doesn’t remain exact with a zero on the right.

izero

6.5.7. Proposition Let $Y \xrightarrow{i} X$ be the inclusion of a closed subvariety Y into a variety X , and let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal of Y . An \mathcal{O}_X -module \mathcal{M} is isomorphic to the direct image $i_*\mathcal{N}$ of an \mathcal{O}_Y -module if and only if \mathcal{I} annihilates \mathcal{M} .

proof. It suffices to prove this when X is an affine variety, say $X = \text{Spec } A$. Let $I = \mathcal{I}(X)$, so that $Y = \text{Spec } B$, where $B = A/I$, and let $M = \mathcal{M}(X)$. The assertion becomes this: An A -module M has the structure of a B -module if and only if $IM = 0$. \square

includeopen

(6.5.8) inclusion of an open subvariety

When $Y \xrightarrow{f} X$ is the inclusion of an *open* subvariety into X , there is a simple operation that goes in the opposite direction from the direct image. With notation as above, let \mathcal{M} be an \mathcal{O}_X -module. An open subset V of Y is also open in X , so $\mathcal{M}(V)$ is defined. An \mathcal{O}_Y -module $f^*\mathcal{M}$ called the *inverse image* is defined by

$$[f^*\mathcal{M}](V) = \mathcal{M}(V).$$

Thus the \mathcal{O}_Y -module $f^*\mathcal{M}$ is obtained from \mathcal{M} by looking only at the open sets that are contained in Y . The sheaf and coherence properties are obvious.

The inverse image operation seems to be trivial, but the next example shows that it is interesting.

fstarforloc

6.5.9. Example. Let X be the affine variety $\text{Spec } A$, let Y be a localization $Y = X_s = \text{Spec } A_s$, and let $Y \xrightarrow{f} X$ be the inclusion map. The \mathcal{O}_X -module \mathcal{M} is determined by its global sections, the A -module $M = \mathcal{M}(X)$, and an \mathcal{O}_Y -module \mathcal{N} is determined by its global sections, an A_s -module $N = \mathcal{N}(Y)$. Here $[f_*\mathcal{N}](X) = N_A$, is the module obtained from the A_s -module N by restricting scalars to A , while $[f^*\mathcal{M}](Y) = [f_*\mathcal{N}](X_s) = \mathcal{M}(X_s)$ is the localization M_s of M . Thus $[f_*f^*]\mathcal{M}(X)$ is the A -module obtained from the localization M_s by restricting scalars to A . \square

One can describe $f_*f^*\mathcal{M}$ as the \mathcal{O}_X -module obtained from \mathcal{M} by allowing coefficients that have poles on the complement of Y .

maptof-
staffstar

6.5.10. Proposition. Let $Y \xrightarrow{f} X$ be the inclusion of an open subvariety into a variety X , let \mathcal{M} be an \mathcal{O}_X -module, and let \mathcal{N} be an \mathcal{O}_Y -module.

- (i) There is a canonical homomorphism of \mathcal{O}_X -modules $\mathcal{M} \rightarrow f_*f^*\mathcal{M}$.
- (ii) The \mathcal{O}_Y -modules $f_*f_*\mathcal{N} \approx \mathcal{N}$ and \mathcal{N} are canonically isomorphic.
- (iii) The functors f_* and f^* are adjoint: Homomorphisms of \mathcal{O}_Y -modules $f^*\mathcal{M} \rightarrow \mathcal{N}$ correspond bijectively to homomorphisms of \mathcal{O}_X -modules $\mathcal{M} \rightarrow f_*\mathcal{N}$.

proof. (i) We must give homomorphisms of $\mathcal{O}_X(U)$ -modules $\mathcal{M}(U) \rightarrow [f_*f^*\mathcal{M}](U)$ for every open subset U of X , compatibly with restrictions to smaller open sets. Let $V = U \cap Y (= f^{-1}U)$. Then $V \subset U$, so we have a restriction map $\mathcal{M}(U) \rightarrow \mathcal{M}(V)$. The required map is the composition

maptofs-
tarfstar

$$(6.5.11) \quad \mathcal{M}(U) \longrightarrow \mathcal{M}(V) \stackrel{def}{=} [f^*\mathcal{M}](V) \stackrel{def}{=} [f_*f^*\mathcal{M}](U)$$

(ii) Let V be an open subset of Y . Then

$$[f_*f_*\mathcal{N}](V) \stackrel{def}{=} [f_*\mathcal{N}](V) \stackrel{def}{=} \mathcal{N}(V \cap Y) = \mathcal{N}(V)$$

(iii) Suppose a homomorphism $\mathcal{M} \xrightarrow{\varphi} f_*\mathcal{N}$ is given. If U is an open subset of X , then $[f_*\mathcal{N}](U) = \mathcal{N}(U \cap Y)$, so φ defines a map $\mathcal{M}(U) \rightarrow \mathcal{N}(U \cap Y)$. Then if V is open in Y , φ give us a map $[f^*\mathcal{M}](V) = \mathcal{M}(V) \rightarrow \mathcal{N}(V \cap Y) = \mathcal{N}(V)$. This is the homomorphism ψ on V . Conversely, suppose that a homomorphism $f^*\mathcal{M} \xrightarrow{\psi} \mathcal{N}$ is given. If U is an open subset of X , we need to find a map $\mathcal{M}(U) \rightarrow f_*\mathcal{N}(U)$. Let $V = U \cap Y$. So V is a subset of U . The required map is the composition

$$\mathcal{M}(U) \rightarrow \mathcal{M}(V) = f^*\mathcal{M}(V) \xrightarrow{\psi} \mathcal{N}(V) = [f_*\mathcal{N}](U)$$

\square

exampledi-
rectimage

6.5.12. Example. Let $X = \mathbb{P}^n$, let f denote the inclusion $\mathbb{U}^0 \subset X$ of the standard affine open subset into X . The direct image $f_*\mathcal{O}_{\mathbb{U}^0}$ of the structure sheaf on Y is an \mathcal{O}_X -module whose sections are rational functions that are allowed to have poles along the hyperplane H at infinity. For example, the sections of $f_*\mathcal{O}_{\mathbb{U}^0}$ on the standard affine $\mathbb{U}^1 : \{x_1 \neq 0\}$ are the rational functions that are regular on $\mathbb{U}^0 \cap \mathbb{U}^1$. They are the homogeneous fractions of degree zero of the form $g(x)/x_0^i x_1^j$. \square

Note. An inverse image functor f^* that is adjoint to f_* can be defined for any morphism of varieties, but its description involves a limit and a tensor product. We'll use the inverse image only for the trivial case of an inclusion of an open subvariety.

directlimits (6.5.13) **direct limits**

A directed set M_\bullet is a sequence of maps

dirset (6.5.14)
$$M_0 \rightarrow M_1 \rightarrow M_2 \rightarrow \dots$$

It could be a sequence of maps of sets, but we'll be interested in directed sets of rings or modules.

A directed set M_\bullet has a *direct limit* $\varinjlim M_\bullet$, which is a set of equivalence classes on the union $\bigcup M_k$. Elements $m \in M_r$ and $m' \in M_s$ are called equivalent if their images in M_n are equal for sufficiently large n . So an element of $\varinjlim M_\bullet$ will be represented by an element of M_k for some k , and elements $m \in M_r$ and $m' \in M_s$ represent the same element of the limit if their images in M_n are equal for large n .

examplelimit **6.5.15. Example.** An increasing family $M_0 \subset M_1 \subset \dots$ of submodules of an R -module S is a directed set whose limit is the union $\bigcup M_k$. The quotients $\overline{S}_n = S/M_n$ of an increasing family of submodules form a directed set whose maps are surjective, and whose limit is $S/(\bigcup M_k)$. \square

limitmodule **6.5.16. Lemma** Let (6.5.14) be a directed of rings or of modules over a ring A . With the structure inherited from the modules M_n , the direct limit $\varinjlim M_\bullet$ is a ring or a module. \square

A homomorphism $M_\bullet \xrightarrow{\varphi} M'_\bullet$ of directed sets of R -modules consists of homomorphisms $M_n \xrightarrow{\varphi_n} M'_n$ for each n that make a (commutative) diagram

$$\begin{array}{ccccccc} \dots & \longrightarrow & M_n & \longrightarrow & M_{n+1} & \longrightarrow & \dots \\ & & \varphi_n \downarrow & & \varphi_{n+1} \downarrow & & \\ \dots & \longrightarrow & M'_n & \longrightarrow & M'_{n+1} & \longrightarrow & \dots \end{array}$$

A sequence $\rightarrow M_\bullet \rightarrow M'_\bullet \rightarrow M''_\bullet \rightarrow$ of such homomorphisms is *exact* if the sequence $\rightarrow M_n \rightarrow M'_n \rightarrow M''_n \rightarrow$ is exact for every n .

limitmodule **6.5.17. Proposition (i)** A homomorphism of directed sets induces a homomorphism of limits.
(ii) If a sequence $M_\bullet \rightarrow M'_\bullet \rightarrow M''_\bullet$ of homomorphisms of directed sets is exact, then the limit sequence $\varinjlim M_\bullet \rightarrow \varinjlim M'_\bullet \rightarrow \varinjlim M''_\bullet$ is exact.
(iii) If M_\bullet is a directed set of R -modules and N is another R -module, then $\varinjlim (M_\bullet \otimes_R N)$ is canonically isomorphic to $(\varinjlim M_\bullet) \otimes_R N$. \square

Let X be a variety, and let $M_\bullet = \{M_0 \rightarrow M_1 \rightarrow M_2 \rightarrow \dots\}$ be a directed set of \mathcal{O}_X -modules. The limit $\varinjlim M_\bullet$ is defined by $[\varinjlim M_\bullet](U) = \varinjlim [M_\bullet(U)]$, for every open set U .

limitsheafgain **6.5.18. Proposition.** Let M_\bullet be a directed set of \mathcal{O} -modules.
(i) The direct limit M_\bullet is an \mathcal{O} -module.
(ii) Direct and inverse images are compatible with limits: $\varinjlim (f_* M_\bullet) \approx f_*(\varinjlim M_\bullet)$ and $\varinjlim (f^* M_\bullet) \approx f^*(\varinjlim M_\bullet)$.
(iii) If \mathcal{N} is an \mathcal{O} -module, then $(\varinjlim M_\bullet) \otimes_{\mathcal{O}} \mathcal{N}$ is isomorphic to $\varinjlim (M_\bullet \otimes_{\mathcal{O}} \mathcal{N})$ \square

6.6 Twisting

(6.6.1) tensor products

Tensor products of modules over a ring are defined in the background chapter. We have suppressed their use so far, but now is the time to learn the definition.

Tensor products are compatible with localization: If M and N are modules over a domain A and s is a nonzero element of A , then $(M \otimes_A N)_s$ and $M_s \otimes_{A_s} N_s$ are isomorphic. This allows us to extend tensor products to \mathcal{O} -modules. The *tensor product* is defined by the standard procedure. If U is an affine open set, then

$$[\mathcal{M} \otimes_{\mathcal{O}} \mathcal{N}](U) \stackrel{def}{=} \mathcal{M}(U) \otimes_{\mathcal{O}(U)} \mathcal{N}(U)$$

Because this tensor product is over the structure sheaf \mathcal{O} , functions move through the tensor symbol: If α is a function, then $m\alpha \otimes n = m \otimes \alpha n$.

Various canonical isomorphisms, such as the isomorphism $\mathcal{M} \otimes_{\mathcal{O}} \mathcal{O} \approx \mathcal{M}$ given by $m \otimes 1 \rightsquigarrow m$, carry over from A -modules to \mathcal{O} -modules.

6.6.2. Lemma. *For any open set V there is a canonical map of $\mathcal{O}(V)$ -modules*

$$\mathcal{M}(V) \otimes_{\mathcal{O}(V)} \mathcal{N}(V) \xrightarrow{\epsilon} [\mathcal{M} \otimes_{\mathcal{O}} \mathcal{N}](V)$$

proof. Let $\{V^i\}$ be an affine covering of V . We need some notation: Let $R_i = \mathcal{O}(U^i)$, $M_i = \mathcal{M}(U^i)$, $N_i = \mathcal{N}(U^i)$, and $T^i = [\mathcal{M} \otimes_{\mathcal{O}} \mathcal{N}](U^i)$, and let R_{ij}, M_{ij}, N_{ij} and T_{ij} be defined analogously, for sections on the intersections $U^i \cap U^j$. Then

$$\mathcal{M}(V^i) \otimes_{\mathcal{O}(V^i)} \mathcal{N}(V^i) \approx M_i \otimes_{R_i} N_i \approx T_i$$

We form a diagram

$$\begin{array}{ccccccc} \mathcal{M}(V) \otimes_{\mathcal{O}(V)} \mathcal{N}(V) & \longrightarrow & \prod M_i \otimes_{R_i} N_i & \longrightarrow & \prod M_{ij} \otimes_{R_{ij}} N_{ij} & & \\ \epsilon \downarrow & & \parallel & & \parallel & & \\ 0 & \longrightarrow & [\mathcal{M} \otimes_{\mathcal{O}} \mathcal{N}](V) & \longrightarrow & \prod T_i & \longrightarrow & \prod T_{ij} \end{array}$$

in which the bottom row is exact. The map ϵ is induced from the rest of the diagram. \square

By definition of the tensor product, ϵ is an isomorphism when V is affine. When V isn't affine, the map needn't be injective or surjective.

Example. The multiplication map $\mathcal{O}(r) \times \mathcal{O}(s) \rightarrow \mathcal{O}(r+s)$ of twisting modules on \mathbb{P}^n induces a homomorphism $\mathcal{O}(r) \otimes_{\mathcal{O}} \mathcal{O}(s) \xrightarrow{\varphi} \mathcal{O}(r+s)$. On the standard affine open set \mathbb{U}^0 , the sections of $\mathcal{O}(r) \otimes_{\mathcal{O}} \mathcal{O}(s)$ form a free module with basis $x_0^r \otimes x_0^s$, and this element maps to the basis x_0^{r+s} of $\mathcal{O}(r+s)$ on \mathbb{U}^0 . So φ is an isomorphism.

Let X be the projective line. When $k > 0$, The monomials of degree k form a basis for the space of global sections of $\mathcal{O}(k)$. So $[\mathcal{O}(1)](X)$ has dimension 2. Therefore $[\mathcal{O}(1)](X) \otimes_{\mathcal{O}(X)} [\mathcal{O}(1)](X)$ has dimension 4, whereas the space of sections of the tensor product $\mathcal{O}(1) \otimes_{\mathcal{O}} \mathcal{O}(1) \approx \mathcal{O}(2)$ has dimension 3. The map ϵ isn't injective when $V = X$ and $r, s = 1, 1$. When $r, s = -1, 1$, φ isn't surjective. \square

Let \mathcal{M} be an \mathcal{O} -module on projective space, and let $\mathcal{O}(k)$ denote the twisting sheaf on \mathbb{P} . The k th *twist* of \mathcal{M} is the tensor product

$$(6.6.3) \quad \mathcal{M}(k) = \mathcal{M} \otimes_{\mathcal{O}} \mathcal{O}(k)$$

If \mathcal{M} is an \mathcal{O} -module on a projective variety X , the *twist* $\mathcal{M}(n)$ is the \mathcal{O}_X -module obtained by twisting its extension by zero to \mathbb{P} . (see ??).

imageoff **6.6.4. Lemma.** Let C be the hypersurface in projective space that is defined by an irreducible homogeneous polynomial f , let \mathcal{I} be the ideal of C , and let \mathcal{M} be an $\mathcal{O}_{\mathbb{P}^d}$ -module.

(i) Multiplication by f defines a map

$$\mathcal{M}(k) \xrightarrow{1 \otimes f} \mathcal{M}(k+d)$$

(ii) The image of the map $\mathcal{M}(-d) \xrightarrow{1 \otimes f} \mathcal{M}$ is the submodule $\mathcal{M}\mathcal{I}$ of \mathcal{M} .

See (6.2.13) and (6.2.15).

When possible, we often abbreviate by writing f instead of $a \otimes f$. For example, multiplication by $x_0^d = 1 \otimes x_0^d$ defines a homomorphism

twistmuzero (6.6.5)
$$\mathcal{M}(k) \xrightarrow{x_0^d} \mathcal{M}(k+d)$$

This homomorphism is an isomorphism on \mathbb{U}^0 , where the sections of $\mathcal{O}(d)$ form a free module with basis x_0^d on \mathbb{U}^0 . As for the structure sheaf, this isomorphism is related to the homomorphism $\mathcal{M}(k) \xrightarrow{x_1^d} \mathcal{M}(k+d)$ by the factor $(x_0/x_1)^d$.

generators (6.6.6) **generating a module**

glsechomom **6.6.7. Lemma.** Let \mathcal{M} be an \mathcal{O} -module on a variety X . Global sections of \mathcal{M} correspond bijectively to homomorphisms $\mathcal{O} \rightarrow \mathcal{M}$. If m is a global section, the corresponding homomorphism sends a section α of \mathcal{O} on an open set U to the section αm of \mathcal{M} on U . In the other direction, if $\mathcal{O} \xrightarrow{\varphi} \mathcal{M}$ is a homomorphism, the corresponding global section of \mathcal{M} is $\varphi(1)$. \square

A set $m = (m_1, \dots, m_k)$ of global sections *generates* \mathcal{M} if the map of \mathcal{O} -modules

gen (6.6.8)
$$\mathcal{O}^k \xrightarrow{m} \mathcal{M}$$

that sends $(\alpha_1, \dots, \alpha_k) \rightsquigarrow \alpha_1 m_1 + \dots + \alpha_k m_k$ is surjective. If the sections m generate \mathcal{M} , then they generate the $\mathcal{O}(U)$ -module $\mathcal{M}(U)$ for every affine open set U . They may not generate when U isn't affine, as example (6.5.6) shows. Furthermore, there may not be enough global sections to generate. For instance, it is possible that the only global section is zero.

Let X be the projective space \mathbb{P}^d . The twisting modules $\mathcal{O}(n) = \mathcal{O}_X(n)$ on X can be assembled into a directed set by repeated multiplication by x_0 :

Oninclusions (6.6.9)
$$\mathcal{O} \xrightarrow{x_0} \mathcal{O}(1) \xrightarrow{x_0} \mathcal{O}(2) \xrightarrow{x_0} \dots$$

and multiplication by $1 \otimes x_0$ defines a directed set

cmnsequence (6.6.10)
$$\mathcal{M} \xrightarrow{x_0} \mathcal{M}(1) \xrightarrow{x_0} \mathcal{M}(2) \xrightarrow{x_0} \dots$$

We'll denote the limits of the sequences (6.6.9) and (6.6.10) by $\varinjlim_{x_0} \mathcal{O}(n)$ and $\varinjlim_{x_0} \mathcal{M}(n)$.

Onsequence **6.6.11. Proposition.** Let j denote the inclusion of the standard affine open set \mathbb{U}^0 into \mathbb{P}^d and let \mathcal{M} be an \mathcal{O} -module. Then $\varinjlim_{x_0} \mathcal{M}(n)$ is isomorphic to $j_* j^* \mathcal{M}$.

proof. If V is an affine open set, then $[j_* j^* \mathcal{M}](V) = [j^* \mathcal{M}](V \cap \mathbb{U}^0) = \mathcal{M}(V \cap \mathbb{U}^0)$. The sections of $\mathcal{M}(n) = \mathcal{M} \otimes_{\mathcal{O}} \mathcal{O}(n)$ are sums of tensors $m \otimes u$, where m is in $\mathcal{M}(V)$ and u is a homogeneous fraction of degree n , regular on V . Multiplication by x_0^{-n} defines a map $\mathcal{M}(n) \xrightarrow{x_0^{-n}} j_* j^* \mathcal{M}$ that sends $m \otimes u$ to $m \otimes \alpha$, where $\alpha = u x_0^{-n}$, which is a regular function on $V \cap \mathbb{U}^0$. This gives us a homomorphism of directed sets

$$\begin{array}{ccccc} \xrightarrow{x_0} & \mathcal{M}(n) & \xrightarrow{x_0} & \mathcal{M}(n+1) & \xrightarrow{x_0} \\ & x_0^{-n} \downarrow & & x_0^{-n-1} \downarrow & \\ \xrightarrow{=} & j_* j^* \mathcal{M} & \xrightarrow{=} & j_* j^* \mathcal{M} & \xrightarrow{=} \end{array}$$

which defines a map

$$\varinjlim_{x_0} \mathcal{M}(n) \xrightarrow{\epsilon} j_* j^* \mathcal{M}$$

To show that ϵ is an isomorphism, it suffices to show that ϵ is bijective on sections on the standard affine open sets \mathbb{U}^i . This follows from Corollary ??.

see Proposition 6.3.10). On \mathbb{U}^0 , $\mathcal{M}(n)$ and $j_* j^* \mathcal{M}$ are isomorphic to \mathcal{M} , and multiplication by x_0 is a bijective map. So ϵ is bijective on \mathbb{U}^0 . We look at sections on \mathbb{U}^1 , where $[j_* j^* \mathcal{M}](\mathbb{U}^1) = \mathcal{M}(\mathbb{U}^{01})$. Let $A = \mathcal{O}(\mathbb{U}^1)$ and $M = \mathcal{M}(\mathbb{U}^1)$. The coherence condition for \mathcal{M} tells us that $\mathcal{M}(\mathbb{U}^{01})$ is the localization M_s where $s = x_0/x_1$, and $M_s \approx M \otimes_A A_s$. Recall that $[\mathcal{O}(n)](\mathbb{U}^1)$ is a free A -module with basis x_1^n . So the elements of $[\mathcal{M}(n)](\mathbb{U}^1)$ have the form $m \otimes x_1^n$ with m in M . Multiplication by x_0^{-n} sends $m \otimes x_1^n$ to $m \otimes s^{-n}$. This is the element of $M \otimes_A A_s$ that corresponds to ms^{-n} in M_s . It follows that the map $\varinjlim_{x_0} \mathcal{M}(n) \rightarrow j_* j^* \mathcal{M}$ is surjective on \mathbb{U}^1 . If $m \otimes s^{-n} = 0$, some power s^k of s annihilates m . In that case the image of $m \otimes x_0^n$ in $[\mathcal{M}(n+k)](\mathbb{U}^1)$ is $m \otimes x_1^n x_0^k = m \otimes s^k x_1^{n+k} = ms^k \otimes x_1^{n+k} = 0$. So $m \otimes x_1^n$ is also zero in $\varinjlim_{x_0} \mathcal{M}(n)$. The map ϵ is bijective on \mathbb{U}^1 . \square

genposexact

6.6.12. Corollary. *Let j denote the inclusion of the standard affine open set \mathbb{U}^0 into \mathbb{P}^d .*

(i) *For any \mathcal{O} -module \mathcal{M} , $j_* j^* \mathcal{M}$ is generated by global sections.*

(ii) *Let M be a finite \mathcal{O} -module on a projective variety X . For large n , the twist $\mathcal{M}(n)$ is generated by global sections.*

proof. (i) For any $n \geq 0$, $\mathcal{O}(n)$ is generated by the global sections x_0^n, \dots, x_d^n . Since $\varinjlim_{x_0} \mathcal{O}(n) = j_* j^* \mathcal{O}$, $j_* j^* \mathcal{O}$ is generated by global sections. Next, $M = \mathcal{M}(\mathbb{U}^0)$ is a module over the polynomial algebra $A = \mathcal{O}(\mathbb{U}^0)$. We choose generators for M , obtaining a surjective map $A^N \rightarrow M$, where N may be infinite, and we show that the induced map $(j_* j^* \mathcal{O})^N \rightarrow j_* j^* \mathcal{M}$ is surjective. Then since $j_* j^* \mathcal{O}$ is generated by global sections, so is $j_* j^* \mathcal{M}$.

Let V be an affine open subset of X . Because V and \mathbb{U}^0 are affine, so is $V \cap \mathbb{U}^0$ (3.5.11). By definition, $[j_* j^* \mathcal{O}](V) = \mathcal{O}(V \cap \mathbb{U}^0)$ and $[j_* j^* \mathcal{M}](V) = \mathcal{M}(V \cap \mathbb{U}^0)$. Because $V \cap \mathbb{U}^0$ is affine, the fact that the map $A^N \rightarrow M$ is surjective implies that the restriction of the map $\mathcal{O}^N \rightarrow \mathcal{M}$ to \mathbb{U}^0 is surjective, and therefore that $\mathcal{O}(V \cap \mathbb{U}^0)^N \rightarrow \mathcal{M}(V \cap \mathbb{U}^0)$ is surjective. So $[j_* j^* \mathcal{O}](V)^N \rightarrow [j_* j^* \mathcal{M}](V)$ is surjective, as required.

(ii) We may assume that $X = \mathbb{P}^d$. We look on the standard affine open set \mathbb{U}^0 , where $M = \mathcal{M}(\mathbb{U}^0)$ is a finite module over $A = \mathcal{O}(\mathbb{U}^0)$. Let x_1, \dots, x_k be module generators for M . These generators are also sections of $j_* j^* \mathcal{M}$, and therefore they define a map $\mathcal{O}^k(x) \rightarrow j_* j^* \mathcal{M}$. Then since $j_* j^* \mathcal{M} = \varinjlim_{x_0} \mathcal{M}(n)$, x_i will be in $\mathcal{M}(n)$ for any sufficiently large n , and this will define a map $\mathcal{O}^k(x) \rightarrow \mathcal{M}(n)$. Let \mathcal{C}_n denote the cokernel of this map. Because the elements x_i generate the A -module M , and because $\mathcal{M}(n)(\mathbb{U}^0) = \mathcal{M}(\mathbb{U}^0) = M$, the support of \mathcal{C}_n is contained in the hyperplane H^0 at infinity.

So for large n , the global sections of $\mathcal{M}(n)$ generate a submodule \mathcal{S} large enough so that the cokernel $\mathcal{M}(n)/\mathcal{S} = \mathcal{C}_n$ has support in H^0 . The same reasoning shows that, increasing n as necessary, the support of \mathcal{C}_n will be contained in the hyperplane H^i for any i . Since the intersection of the hyperplanes H^i is empty, $\mathcal{M}(n)$ is generated by global sections. \square

6.7 Proofs

modproofs

We prove the theorems that were stated without proof earlier. One reason that we deferred some proofs to the end of the chapter is that the notation gets complicated. It will help to introduce a symbol such as \mathbb{U} to represent a family $\{U^i\}$ of open sets. A *morphism* from one family $\mathbb{V} = \{V^\nu\}$ of open sets to another family $\mathbb{U} = \{U^i\}$ consists of a morphism (an inclusion), of each V^ν into one of the subsets U^i . Such a morphism will be given by a map of index sets $\nu \rightsquigarrow i(\nu)$, such that V^ν is contained in $U^{i(\nu)}$. There may be more than one morphism $\mathbb{V} \rightarrow \mathbb{U}$, because a subset V^ν may be contained in more than one subset U^i . To define a morphism, one must make a choice among those subsets.

We extend an \mathcal{O} -module to families of open sets by defining

$$(6.7.1) \quad \mathcal{M}(\mathbb{U}) = \prod \mathcal{M}(U^i).$$

sectionson-family

Then a morphism $\mathbb{V} \xrightarrow{f} \mathbb{U}$ of families defines a map $\mathcal{M}(\mathbb{U}) \xrightarrow{f^*} \mathcal{M}(\mathbb{V})$ in a way that is fairly obvious, though our notation for it is clumsy. A section of $\mathcal{M}(\mathbb{U})$ can be thought of as a vector (u_i) with $u_i \in \mathcal{M}(U^i)$. Its restriction to \mathbb{V} is (v_ν) , where v_ν is the restriction of $u_{i(\nu)}$ to V^ν , $i(\nu)$ being the map of indices as above.

Let \mathbb{U}_0 denote a family $\{U^i\}$ of open subsets that covers another open set U , and let \mathbb{U}_1 denote the family of intersections $\{U^i \cap U^j\}$, indexed by ordered pairs i, j of indices. The inclusions $U^i \cap U^j \rightarrow U^j$ and $U^i \cap U^j \rightarrow U^i$ define morphisms of families $\mathbb{U}_1 \rightrightarrows \mathbb{U}_0$. We call the diagram that we obtain,

$$\text{covdiagram} \quad (6.7.2) \quad U \longleftarrow \mathbb{U}_0 \rightrightarrows \mathbb{U}_1$$

a *covering diagram*.

With this notation, the exact sequence (6.2.17) that expresses the sheaf property for the covering becomes

$$\text{sheafone} \quad (6.7.3) \quad 0 \rightarrow \mathcal{M}(U) \xrightarrow{\alpha} \mathcal{M}(\mathbb{U}_0) \xrightarrow{\beta} \mathcal{M}(\mathbb{U}_1)$$

in which the maps α and β are as in (6.2.17).

Proof of Theorem 6.3.1 (about the coherence property)

We are given an \mathcal{O} -module \mathcal{M} that has the sheaf property, and that has the coherence property for the open sets U^i of an affine covering of the variety X . We must prove the coherence property for an arbitrary open set V . Since the coherence property is true on U^i , it is also true on any localization of U^i , and we can cover V by such localizations, say by the affine varieties $V^\nu = \text{Spec } A_\nu$. We relabel V as X and $\{V^\nu\}$ as $\{U^i\}$. The statement to be proved becomes this: Suppose that the coherence property for \mathcal{M} is true for the open sets of an affine covering $\mathbb{U}_0 = \{U^i\}$ of X . If $\mathcal{O}(X) = R$ and $\mathcal{M}(X)$ is the R -module M , then for every nonzero element s of R , $\mathcal{M}(X_s)$ is the localization M_s . In particular, $\mathcal{O}(X_s) = R_s$. What is clear here is that, since s doesn't vanish on X_s , its inverse is in $\mathcal{O}(X_s)$. Therefore $R_s \subset \mathcal{O}(X_s)$.

Let \mathbb{U}_1 denote the family of intersections $U^i \cap U^j$. Since \mathcal{M} has the sheaf property, the sequence

$$\text{covoneagain} \quad (6.7.4) \quad 0 \rightarrow \mathcal{M}(X) \xrightarrow{\alpha} \mathcal{M}(\mathbb{U}_0) \xrightarrow{\beta} \mathcal{M}(\mathbb{U}_1)$$

is exact. We may regard this as a sequence of R -modules by restriction of scalars. Since localization is an exact operation, the localized sequence of R_s -modules

$$\text{covtwo} \quad (6.7.5) \quad 0 \rightarrow \mathcal{M}(X)_s \rightarrow \mathcal{M}(\mathbb{U}_0)_s \rightarrow \mathcal{M}(\mathbb{U}_1)_s$$

is exact. The family of localizations $\mathbb{U}_{0,s} = \{U_s^i\}$ is an affine covering of X_s , and $\mathbb{U}_{1,s}$ is the family of intersections $\{U_s^i \cap U_s^j\}$. Then

$$\text{covthree} \quad (6.7.6) \quad 0 \rightarrow \mathcal{M}(X_s) \rightarrow \mathcal{M}(\mathbb{U}_{0,s}) \rightarrow \mathcal{M}(\mathbb{U}_{1,s})$$

is an exact sequence that can be made into a sequence of R_s -modules by restriction of scalars. Since \mathcal{M} is a functor, (6.7.4) maps to (6.7.6), and since (6.7.6) is a sequence of R_s -modules, the sequence (6.7.5) maps to (6.7.6):

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{M}(X)_s & \longrightarrow & \mathcal{M}(\mathbb{U}_0)_s & \longrightarrow & \mathcal{M}(\mathbb{U}_1)_s \\ & & a \downarrow & & b \downarrow & & c \downarrow \\ 0 & \longrightarrow & \mathcal{M}(X_s) & \longrightarrow & \mathcal{M}(\mathbb{U}_{0,s}) & \longrightarrow & \mathcal{M}(\mathbb{U}_{1,s}) \end{array}$$

The coherence property for the covering \mathbb{U}_0 implies that the map b is bijective. Therefore the map a is injective, and since X was arbitrary, this is also true for every open subset V of X . Therefore the map c is injective. Then it follows from the diagram that a is bijective, as the coherence property asserts. \square

Proof of Theorem 6.3.5 (extending a module given on a basis)

We are given a good basis \mathcal{B} for the topology on the variety X , and a quasicohherent \mathcal{B} -module. We must extend \mathcal{M} to an \mathcal{O} -module on X . The proof isn't difficult, but there are several things to check.

We call an open set that is a member of \mathcal{B} a \mathcal{B} -open set, and a covering of any open set by \mathcal{B} -open sets a \mathcal{B} -covering.

Let \mathcal{M} be a \mathcal{B} -module. In this proof, we denote the \mathcal{O} -module that is to be constructed by $\widetilde{\mathcal{M}}$, though when we are done, we will use the same symbol for a \mathcal{B} -module and for the \mathcal{O} -module on X that it determines. Suppose given a \mathcal{B} -covering $\mathbb{U}_0 = \{U^i\}$ of an open subset Y . The \mathcal{O} -module $\widetilde{\mathcal{M}}$ that we are looking for must have the property that $\widetilde{\mathcal{M}}(\mathbb{U}_0) = \mathcal{M}(\mathbb{U}_0)$ and $\widetilde{\mathcal{M}}(\mathbb{U}_1) = \mathcal{M}(\mathbb{U}_1)$. So the sheaf property for $\widetilde{\mathcal{M}}$ becomes the exactness of the sequence

$$\text{defm} \quad (6.7.7) \quad 0 \rightarrow \widetilde{\mathcal{M}}(Y) \xrightarrow{\alpha} \mathcal{M}(\mathbb{U}_0) \xrightarrow{\beta} \mathcal{M}(\mathbb{U}_1)$$

This identifies $\widetilde{\mathcal{M}}(Y)$ as a kernel, but there will be many \mathcal{B} -coverings of a given open set. We show first that all such kernels are isomorphic.

6.7.8. Lemma. *Let $\mathbb{U}_0 = \{U^i\}$ and $\mathbb{V}_0 = \{V^\nu\}$ be \mathcal{B} -coverings of an open set Y . There is a canonical isomorphism*

$$\ker(\mathcal{M}(\mathbb{U}_0) \xrightarrow{\beta} \mathcal{M}(\mathbb{U}_1)) \approx \ker(\mathcal{M}(\mathbb{V}_0) \xrightarrow{\beta'} \mathcal{M}(\mathbb{V}_1)).$$

proof. Let $\mathbb{U}_0\mathbb{V}_0$ denote the family of intersections $\{U^i \cap V^\nu\}$, indexed by pairs i, ν of indices, let $\mathbb{U}_1\mathbb{V}_0$ denote the family $\{U^i \cap U^j \cap V^\nu\}$, indexed by triples i, j, ν of indices, and let $\mathbb{U}_0\mathbb{V}_1$ be defined analogously. We form a diagram

(The arrows with an asterisk in the diagram below are supposed to be double arrows.)

$$\text{comdiag} \quad (6.7.9) \quad \begin{array}{ccccc} \mathbb{U}_1 & \longleftarrow & \mathbb{U}_1\mathbb{V}_0 & & \\ \downarrow^* & & \downarrow^* & & \\ \mathbb{U}_0 & \longleftarrow & \mathbb{U}_0\mathbb{V}_0 & \xleftarrow{*} & \mathbb{U}_0\mathbb{V}_1 \\ \downarrow & & \downarrow & & \downarrow \\ Y & \longleftarrow & \mathbb{V}_0 & \xleftarrow{*} & \mathbb{V}_1 \end{array}$$

Except for Y , all sets in this diagram are families of \mathcal{B} -open sets. The bottom row and the left column are covering diagrams, the middle row and middle column are families of covering diagrams, and the top horizontal arrow and the right vertical arrow are families of coverings.

We apply the \mathcal{B} -module to the diagram, omitting Y :

$$\text{comdtwo} \quad (6.7.10) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{M}(\mathbb{U}_1) & \longrightarrow & \mathcal{M}(\mathbb{U}_1\mathbb{V}_0) & & \\ & & \uparrow^{\beta} & & \uparrow & & \\ 0 & \longrightarrow & \mathcal{M}(\mathbb{U}_0) & \longrightarrow & \mathcal{M}(\mathbb{U}_0\mathbb{V}_0) & \longrightarrow & \mathcal{M}(\mathbb{U}_0\mathbb{V}_1) \\ & & & & \uparrow & & \uparrow \\ & & & & \mathcal{M}(\mathbb{V}_0) & \xrightarrow{\beta'} & \mathcal{M}(\mathbb{V}_1) \\ & & & & \uparrow & & \uparrow \\ & & & & 0 & & 0 \end{array}$$

The sheaf axiom for the \mathcal{B} -module \mathcal{M} implies that all rows and all columns are exact. (Products of exact sequences are exact.) It follows that the kernels of β and β' are canonically isomorphic. \square

The lemma allows us to define $\widetilde{\mathcal{M}}(Y)$ as the kernel of any one of the homomorphisms $\mathcal{M}(\mathbb{U}_0) \rightarrow \mathcal{M}(\mathbb{U}_1)$. We define the $\mathcal{O}(Y)$ -module structure on $\widetilde{\mathcal{M}}(Y)$ next. Let a and m be sections \mathcal{O} and $\widetilde{\mathcal{M}}$ on Y , and let their images in $\mathcal{O}(\mathbb{U}_0)$ and $\mathcal{M}(\mathbb{U}_0)$ be denoted by the same letters. Then am is an element of $\mathcal{M}(\mathbb{U}_0)$. Since $\beta(m)$ is zero, so is $\beta(am)$. Therefore am is an element of $\widetilde{\mathcal{M}}(Y)$. This is the product section. The axioms for a module are easy to verify.

Next, if $Y' \subset Y$ are open sets, there will be \mathcal{B} -coverings \mathbb{U}'_0 of Y' and \mathbb{U}_0 of Y that form a diagram

$$\begin{array}{ccc} \mathbb{U}'_0 & \longrightarrow & \mathbb{U}_0 \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

This diagram will give us a module homomorphism $\ker(\mathcal{M}(\mathbb{U}_0) \rightarrow \mathcal{M}(\mathbb{U}_1)) \rightarrow \ker(\mathcal{M}(\mathbb{U}'_0) \rightarrow \mathcal{M}(\mathbb{U}'_1))$, so $\widetilde{\mathcal{M}}$ will be a functor.

Having defined $\widetilde{\mathcal{M}}(Y)$ for every Y , we must verify the sheaf property for a covering $\mathbb{V}_0 = \{V^\nu\}$ of Y in which the open sets aren't necessarily \mathcal{B} -open. The coherence property will follow from Theorem 6.3.1. To verify the sheaf property, we cover each V^ν by \mathcal{B} -open sets $W^{\nu,i}$, where the index set $\{i\}$ depends on ν . Taken together, the family \mathbb{W}_0 of all of these sets forms a \mathcal{B} -covering of Y that fits into a diagram

$$(6.7.11) \quad \begin{array}{ccccc} Y & \longleftarrow & \mathbb{W}_0 & \xleftarrow{*} & \mathbb{W}_1 \\ \parallel & & \downarrow & & \downarrow \\ Y & \longleftarrow & \mathbb{V}_0 & \xleftarrow{*} & \mathbb{V}_1 \end{array}$$

in which the vertical arrows are families of coverings. Applying $\widetilde{\mathcal{M}}$ gives us a diagram

$$(6.7.12) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \widetilde{\mathcal{M}}(Y) & \longrightarrow & \mathcal{M}(\mathbb{W}_0) & \longrightarrow & \mathcal{M}(\mathbb{W}_1) \\ & & \parallel & & \uparrow & & \uparrow \\ & & \widetilde{\mathcal{M}}(Y) & \longrightarrow & \widetilde{\mathcal{M}}(\mathbb{V}_0) & \xrightarrow{\beta} & \widetilde{\mathcal{M}}(\mathbb{V}_1) \end{array}$$

In this diagram, the top row is exact and the vertical arrows are injective. The fact that $\widetilde{\mathcal{M}}(Y)$ is the kernel of β follows.

It is also true that homomorphisms of \mathcal{B} -modules and of \mathcal{O} -modules correspond bijectively, but we omit the verification. \square

Proof of Theorem 6.3.6 (the sheaf property for an A -module)

We are given a module M over a ring A and nonzero elements s_1, \dots, s_k of A that generate the unit ideal, and we are to prove that when we denote by M_i and M_{ij} the localizations M_{s_i} and $M_{s_i s_j}$, respectively, the sequence

$$0 \rightarrow M \rightarrow \prod M_i \rightarrow \prod M_{ij}$$

is exact.

In this sequence, the map α sends an element m of M to the vector (m, \dots, m) of its images in $M_1 \times \dots \times M_k$, and β sends a vector (m_1, \dots, m_k) in $M_1 \times \dots \times M_k$ to the matrix (z_{ij}) , with $z_{ij} = m_j - m_i$ in M_{ij} .

exactness at M : Suppose that an element m of M maps to zero in every M_i . Then there exists an n such that $s_i^n m = 0$, and we can use the same large exponent n for all i . The elements s_i^n generate the unit ideal. Writing $\sum a_i s_i^n = 1$, we have $m = \sum a_i s_i^n m = \sum a_i 0 = 0$.

exactness at $\prod M_i$: Suppose given elements m_i of M_i and that $m_i = m_j$ in M_{ij} for all i, j . We want to find an element w in M such that $w = m_i$ in M_i for every i . We write $m_i = s_i^{-n} x_i$ with x_i in M , using the same integer n for all i . The equation $m_i = m_j$ in M_{ij} tells us that $s_j^n x_i = s_i^n x_j$ is true in M_{ij} , and then $(s_i s_j)^r s_j^n x_i = (s_i s_j)^r s_i^n x_j$ will be true in M if r is large. We adjust the notation. Let $\tilde{x}_i = s_i^r x_i$, and $\tilde{s}_i = s_i^{r+n}$. Then

$$x_i = \tilde{s}_i^{-1} \tilde{x}_i \quad \text{and} \quad \tilde{s}_j \tilde{x}_i = \tilde{s}_i \tilde{x}_j$$

in M . Since the elements s_i generate the unit ideal, so do the elements \tilde{s}_i . There is an equation of the form $\sum a_i \tilde{s}_i = 1$. Let $w = \sum a_i \tilde{x}_i$. This is an element of M , and

$$\tilde{x}_j = \left(\sum_i a_i \tilde{s}_i \right) \tilde{x}_j = \sum_i a_i \tilde{s}_j \tilde{x}_i = \tilde{s}_j w.$$

Therefore $m_j = \tilde{s}_j^{-1} \tilde{x}_j = w$ is true in M_j . Since j is arbitrary, w is the required element of M . \square

Proof of Proposition 6.3.10 (about finite \mathcal{O} -modules)

We are given an affine cover $U^i = \text{Spec } A_i$ of X and an \mathcal{O} -module \mathcal{M} such that $M_i = \mathcal{M}(U^i)$ is a finite A_i -module for each i , and we are to show that \mathcal{M} is a finite \mathcal{O} -module. This means that for every affine open set $V = \text{Spec } A'$, $M' = \mathcal{M}(V)$ is a finite A' -module. Lemma 3.5.12 tells us that we can cover V by localizations $V^j = \text{Spec } A'_j$, each of which is also a localization of one of the opens U^i . The coherence property shows that $M'_j = \mathcal{M}(V^j)$ will be a localization of one of the finite modules M_i , and therefore it will be a finite A'_j -module. We relabel V as U and drop the primes. What must be proved is this:

Let $U = \text{Spec } A$ and let M be an A -module. Let s_1, \dots, s_k be nonzero elements of A that generate the unit ideal. With notation as in the previous proof, if M_i is a finite A_i -module for every i , then M is a finite A -module.

We choose a finite set $m = \{m_1, \dots, m_r\}$ of elements of M whose images generate the localization M_i for every i . These elements define a homomorphism $A^r \xrightarrow{\varphi} M$ that sends (a_1, \dots, a_r) to $\sum a_i m_i$. Let C be the cokernel of φ . Since m generates M_i and since localization is an exact operation, the localization C_i of C is zero for every i . Theorem 6.3.6, applied to C , shows that $C = 0$. \square

Proof of Theorem 6.4.4 (left exactness of the section functor)

We are given an exact sequence of \mathcal{O} -modules

$$0 \rightarrow \mathcal{M} \rightarrow \mathcal{N} \rightarrow \mathcal{P}$$

We choose a covering diagram $U \leftarrow \mathbb{U}_0 \rightrightarrows \mathbb{U}_1$ in which \mathbb{U}_0 is a family of affine open subsets, and we inspect the diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{M}(U) & \xrightarrow{u} & \mathcal{N}(U) & \xrightarrow{v} & \mathcal{P}(U) \\ & & a \downarrow & & a' \downarrow & & a'' \downarrow \\ 0 & \longrightarrow & \mathcal{M}(\mathbb{U}_0) & \xrightarrow{u'} & \mathcal{N}(\mathbb{U}_0) & \xrightarrow{v'} & \mathcal{P}(\mathbb{U}_0) \\ & & b \downarrow & & b' \downarrow & & b'' \downarrow \\ 0 & \longrightarrow & \mathcal{M}(\mathbb{U}_1) & \xrightarrow{u''} & \mathcal{N}(\mathbb{U}_1) & \xrightarrow{v''} & \mathcal{P}(\mathbb{U}_1) \end{array}$$

The sheaf axiom asserts that the columns of this diagram are exact, and the middle row is exact because \mathbb{U}_0 is a family of affines. We are to show that the top row is exact. The proof is a ‘diagram chase’. We present it, though it is best to make such a verification oneself.

exactness at $\mathcal{M}(U)$: Both u' and a are injective, and $u'a = a'u$. So $a'u$ is injective, and this implies that u is injective. Moreover, this statement is true for every open set U . Therefore u'' is injective too.

exactness at $\mathcal{N}(U)$: The elements to which we will refer are shown in this diagram:

$$\begin{array}{ccc} z & \xrightarrow{u} & x \\ a \downarrow & & a' \downarrow \\ y & \xrightarrow{u'} & a'(x) \end{array}$$

Let x be an element in the kernel of v . We want to show that $x = u(z)$ for some z in $\mathcal{M}(U)$. Since x is in the kernel of v , $a'(x)$ is in the kernel of v' . Since the middle row is exact, $a'(x)$ is the image of an element y of $\mathcal{M}(\mathbb{U}_0)$: $a'(x) = u'(y)$. We also know that $b'a' = 0$, so $b'u'(y) = b'a'(x) = 0$. Since $b'u' = u''b$, we conclude that $u''b(y) = 0$. Since u'' is injective, $b(y) = 0$. Since the first column is exact, $y = a(z)$ for some z in $\mathcal{M}(U)$. Then $a'u(z) = u'a(z) = u'(y) = a'(x)$. Since a' is injective, $u(z) = x$. \square