Chapter 5 STRUCTURE OF VARIETIES IN THE ZARISKI TOPOLOGY

dim

5.1 Dimension

The dimension of a variety can be defined either as a transcendence degree or as the maximal length of a chain of closed subvarieties (see **5.1.3**). We use transcendence degree to define the dimension, and we show that it gives the same answer as the one that would be obtained with chains of subvarieties.

The *dimension* of a variety X is the transcendence degree over $\mathbb{C}$ of its function field, and the dimension of a finite-type domain A is the transcendence degree of its fraction field. Thus if $X = \operatorname{Spec} A$, then

$$\dim X \;=\; \dim A$$

dimequal **5.1.1. Corollary.** *If* $Y \xrightarrow{u} X$ *is the inclusion of an open subvariety or if it is an integral morphism, then* $\dim Y = \dim X$. □

If C is a proper closed subvariety of an affine variety X, some regular function on X will vanish on C. Because of this, C will have lower dimension than X. But it isn't obvious how much lower its dimension will be. A subtle fact known as Krull's Theorem helps to determine the drop in dimension.

The *codimension* of a closed subvariety C of a variety X is the difference $\dim X - \dim C$ of their dimensions.

krullthm **5.1.2. Krull's Principal Ideal Theorem.** *Let* $X = \operatorname{Spec} A$ *be an affine variety of dimension* n*, and let* f *be a nonzero element of* A*. Every irreducible component of the zero locus* $V_X(f)$ *has codimension* 1.

proof. Step 1: The case of affine space. Let A be the polynomial ring $\mathbb{C}[x_1, ..., x_n]$, let $X = \operatorname{Spec} A$, and let f be a nonzero element of A. When we factor f into irreducible polynomials, say $f = f_1 \cdots f_k$, then because $\mathbb{C}[x]$ is a unique factorization domain, the ideals (f_i) will be prime ideals. The irreducible components of the zero locus $V_X(f)$ will be the zero sets $V_X(f_i)$. We may assume that f is irreducible.

We adjust coordinates so that f becomes a monic polynomial in x_n with coefficients in $\mathbb{C}[x_1, \dots, x_{n-1}]$, say $f = x_n^k + c_{k-1}x_n^{k-1} + \cdots + c_0$ (Lemma 4.2.8). Then $\overline{A} = A/(f)$ will be integral over $\mathbb{C}[x_1, \dots, x_{n-1}]$, so it will have transcendence degree $n-1$, and $V_X(f) = \operatorname{Spec} \overline{A}$ will have codimension 1.

We now consider the general case. We suppose that $V_X(f)$ has a component D of dimension k, and we show that $k = n - 1$.

Step 2: Let Z be the union of the components of $V_X(f)$ distinct from D. We eliminate Z by localizing. We choose an element s in A that is identically zero on Z, but not identically zero on D. Then the localization X_s contains points of D, but no point of Z. The dimensions of the localizations X_s and D_s will be the same as the dimensions of X and D. We replace X by X_s and D by D_s.

Step 3: We suppose $D = V_X(f)$ is irreducible, and that it has dimension k, and we apply the Noether Normalization Theorem. There is a polynomial subring $R = \mathbb{C}[x_1, ..., x_n]$ over which A is a finite module. Let F and K be the fraction fields of R and A, respectively. Then K is a finite extension of F that we may embed into a Galois extension K_1 of F. Let A_1 be the integral closure of A in K_1. Then A_1 is also the integral closure of R in K_1. Let $S = \operatorname{Spec} R$, $X_1 = \operatorname{Spec} A_1$, and let W_1 be the zero loci of f in X_1. Since the morphism $X_1 \to X$ is integral, W_1 maps surjectively to D. Every component D_1 of W_1 lies over a subvariety of D, though not necessarily over D itself. So every component of W_1 will have dimension at most k, and at least one component will have dimension equal to k. We replace A, X, and D by A_1, X_1, and W_1. The set W_1 isn't necessarily irreducible, but the important point is that all of its components have dimension at most k, and that at least one has dimension k. So we may assume that K is a Galois extension of F. We drop the subscript 1.

Step 4: Let G be the Galois group of K over F, and let $f_1, ..., f_r$ be the G-orbit of f, with $f = f_1$. The elements f_i are integral over R, and the product $g = f_1 \cdots f_r$ is in F. Since R is integrally closed, g is in R. We will show that the zero locus $V = V_S(g)$ is the image of $W = V_X(f)$. Then since X is integral over S, every component C of V will be the image of a component D of W, and the dimensions of C and D will be equal. According to Step 1, every component of V has codimension 1. Therefore D and C have dimension $n-1$. This will show that $k = n-1$.

Let p be a point of V, and let q be a point of X that lies over p. Then $g(q) = g(p) = 0$. Since $g = f_1 \cdots f_r$, $f_i(q) = 0$ for some i. The elements f_i form an orbit, so $f_i = \sigma f_1$ for some σ in G. Let π_q denote the homomorphism $A \to \mathbb{C}$ that corresponds to q, as usual. Then (2.8.2)

$$0 = f_i(q) = \pi_q(f_i) = \pi_q(\sigma f_1) = \pi_{q\sigma}(f_1) = f_1(q\sigma)$$

Since q lies over p, so does $q' = q\sigma$. Since $f(q') = 0$, q' is in W. Therefore p is in the image of W. □

chains **(5.1.3) chains of subvarieties**

A *chain* of subvarieties of X *of length* k is a strictly decreasing sequence

chntwo (5.1.4)
$$C_0 > C_1 > C_2 > \cdots > C_k$$

of closed subvarieties. This chain is *maximal* if it cannot be lengthened by inserting another closed subvariety, which means that $C_0 = X$, that for $i < k$ there is no closed subvariety $\widetilde{C}$ with $C_i > \widetilde{C} > C_{i+1}$, and that C_k is a point.

Maximal chains in $\mathbb{P}^2$ have the form $\mathbb{P}^2 > C > p$, where C is a plane curve and p is a point. The chain

$$\mathbb{P}^n > \mathbb{P}^{n-1} > \cdots > \mathbb{P}^0$$

in which $\mathbb{P}^k$ is the set of points $(x_0, ..., x_k, 0, ..., 0)$ of $\mathbb{P}^n$ is a maximal chain of closed subvarieties of $\mathbb{P}^n$.

restrictchain **5.1.5. Lemma.** *Let X' be an open subvariety of a variety X. There is a bijective correspondence between chains $C_0 > \cdots > C_k$ of closed subvarieties of X such that $C_k \cap X' \neq \emptyset$ and chains $C'_0 > \cdots > C'_k$ of closed subvarieties of X', defined by $C'_i = C_i \cap X'$. Given a chain C'_i in X', the corresponding chain in X consists of the closures C_i of the varieties C'_i in X.*

proof. Suppose given a chain $\{C_i\}$ and that $C_k \cap X' \neq \emptyset$. Then the intersections $C'_i = C_i \cap X'$ are nonempty for all i, so they are dense open subsets of the irreducible closed sets C_i (2.7.6). The closure of C'_i is C_i. Since C_i is irreducible and $C_i > C_{i+1}$, it is also true that C'_i is irreducible and $C'_i > C'_{i+1}$. Therefore $C'_0 > \cdots > C'_k$ is a chain of closed subsets of X'. Conversely, if $C'_0 > \cdots > C'_k$ is a chain in X', the closures in X form a chain in X (see Corollary 2.7.8). □

codimdim **5.1.6. Lemma.** *A closed subvariety C of a variety X has codimension* 1 *if and only if $X > C$ and there is no closed subvariety $\widetilde{C}$ such that $X > \widetilde{C} > C$.*

proof. Say that $\dim X = n$. As Lemma 5.1.5 shows, we may assume X affine. We may also assume that $X > C$. Then there will be a regular nonzero function f that vanishes on C. Since C is irreducible, it will be contained in a component $\widetilde{C}$ of the zero locus of f, and by Krull's Theorem, $\widetilde{C}$ will have codimension 1. If C has codimension greater than 1, then $X > \widetilde{C} > C$. For the converse, suppose that there is a closed subvariety $\widetilde{C}$ of X such that $X > \widetilde{C} > C$. Then $\widetilde{C}$ will have codimension at least 1. We apply Krull's Theorem to $\widetilde{C}$. There will be a nonzero regular function g on $\widetilde{C}$ that vanishes on C, and then C will be contained in a component of the zero locus of g, which will have codimension 1 in $\widetilde{C}$. Then C will have codimension at least 2 in X. □

contcodimone **5.1.7. Corollary.** *Every proper closed subvariety of a variety X is contained in a closed subvariety of codimension* 1. □

dimtheorem **5.1.8. Theorem.** *Let X be a variety of dimension n. All chains of closed subvarieties of X have length at most n, and all maximal chains have length n.*

proof. Induction allows us to assume the theorem true for a variety of dimension less than n, and the case $n = 0$ is trivial.

Let X be a variety of dimension n. Lemma 5.1.5 shows that we may assume X affine, say $X = \operatorname{Spec} A$. Let $C_0 > C_1 > \cdots > C_k$ be a chain in X. We are to show that $k \le n$ and that $k = n$ if the chain is maximal. We can insert closed subvarieties into the chain where possible, so we may assume that $C_0 = X$ and that C_1 has codimension 1, and dimension $n - 1$.

By induction, the length of the chain $C_1 > \cdots > C_k$, which is $k - 1$, is at most $n - 1$, and is equal to $n - 1$ if it is a maximal chain in C_1. Lemma 5.1.6 shows that this happens if and only if the given chain $C_0 > C_1 > \cdots > C_k$ is maximal in X. □

dimless **5.1.9. Corollary.** *If Y is a proper closed subvariety of a variety X, then* $\dim Y < \dim X$. □

Theorem 5.1.8 can also be stated in terms of prime ideals. A chain (5.1.4) in $X = \operatorname{Spec} A$ will correspond to an increasing chain

chn **(5.1.10)**
$$P_0 < P_1 < P_2 < \cdots < P_k,$$

of prime ideals of A of length k, a *prime chain*. This prime chain is *maximal* if it cannot be lengthened by inserting another prime ideal, which means that P_0 is the zero ideal, that for $i < k$ there is no prime ideal $\widetilde{P}$ with $P_i < \widetilde{P} < P_{i+1}$, and that P_k is a maximal ideal. In terms of prime chains, Theorem 5.1.8 is this:

dimtheoremtwo **5.1.11. Corollary.** *Let A be a finite-type domain of transcendence degree n. All prime chains in A have length at most n, and all maximal prime chains have length equal to n.* □

For example, the polynomial algebra $\mathbb{C}[x_1, \dots, x_n]$ in n variables has transcendence degree n, and therefore it has dimension n. The chain of prime ideals

primechain **(5.1.12)**
$$0 < (x_1) < (x_1, x_2) < \cdots < (x_1, \dots, x_n)$$

is a maximal prime chain.

A prime ideal P of a noetherian domain has *codimension* 1 if it is not the zero ideal, and if there is no prime ideal $\widetilde{P}$ such that $(0) < \widetilde{P} < P$. Krull's Theorem shows that the prime ideals of codimension 1 in the polynomial algebra $\mathbb{C}[x_1, \dots, x_n]$ are the principal ideals generated by irreducible polynomials.

5.2 Localization II

locring

If s is a nonzero element of a domain A, the simple localization A_s is the ring obtained by adjoining an inverse of a nonzero element s. To work with the inverses of finitely many nonzero elements, one may simply adjoin the inverse of their product.

For working with an infinite set of inverses, the concept of a multiplicative system is useful. A *multiplicative system S* in a domain A is a subset that consists of nonzero elements, is closed under multiplication, and contains 1. If S is a multiplicative system, the ring of *S-fractions* AS^{-1}. It is also called a *localization* of A. This localization is the ring obtained by inverting all elements of S. Its elements are equivalence classes of fractions as^{-1} with a in A and s in S, the equivalence relation and the laws of composition being the usual ones for fractions.

inverseex-amples

5.2.1. Examples. **(i)** The set consisting of the powers of a nonzero element s is a multiplicative system. The ring of fractions of this system is the simple localization $A_s = A[s^{-1}]$.

(ii) When S is the set of all nonzero elements of A, the localization AS^{-1} is the field of fractions of A.

(iii) Let P be a prime ideal of A. The complement of P in A is a multiplicative system.

If s_1 and s_2 aren't in P, then because P is a prime ideal, the product s_1s_2 isn't in P either. The unit element 1 isn't in P because P isn't the unit ideal. In fact, an ideal is a prime ideal if and only if its complement is a multiplicative system. □

extendide-altoloc

5.2.2. Proposition. *Let S be a multiplicative system in a domain A, and let A' denote the localization AS^{-1}.*
(i) *Let I be an ideal of A. The extended ideal IA' is the set IS^{-1} whose elements are classes of fractions xs^{-1}, with x in I and s in S. The extended ideal is the unit ideal if and only if I contains an element of S.*
(ii) *Let J be an ideal of A' and let I denote its contraction $J \cap A$. The extended ideal IA' is equal to J:*
(iii) *If P is a prime ideal of A and if $P \cap S$ is empty, the extended ideal $P' = PA'$ is a prime ideal of A', and its contraction $P' \cap A$ is P. If $P \cap S$ isn't empty, the extended ideal is the unit ideal.* □

Thus $J = extend(contract(J))$, and $I \subset contract(extend(I))$.

Part **(iii)** tells us that prime ideals of A' correspond bijectively to prime ideals of A that don't meet S.

locfintype

5.2.3. Corollary. *A localization AS^{-1} of a noetherian domain A is noetherian.* □

importprinc

5.2.4. Note. An elementary, but important, principle for working with fractions is that any finite sequence of computations in a localization AS^{-1} will involve only finitely many denominators, and can therefore be done in a simple localization A_s, where s is a common denominator for the fractions that occur. The next proposition makes use of this principle.

localnthe-orem

5.2.5. Proposition. *Let $A \subset B$ be finite-type domains. There is a nonzero element s in A such that B_s is a finite module over a subring of the form $A_s[y_1, ..., y_r]$, whose elements are polynomials with coefficients in A_s.*

proof. Let S be the set of nonzero elements of A, so that AS^{-1} is the fraction field K of A, and let $B_K = BS^{-1}$. Then B_K is a finite-type K-algebra. It is generated as K-algebra by a set $\beta_1, ..., \beta_r$ that generates the finite-type $\mathbb{C}$-algebra B. The Noether Normalization Theorem tells us that B_K is a finite module over a polynomial subring $K[y_1, ..., y_r]$. Then B is an integral extension of this polynomial ring.

Any element b of B will be in B_K, and therefore it will be the root of a monic polynomial of the form

$$f(x) = x^n + c_{n-1}(y)x^{n-1} + \cdots + c_0(y) = 0$$

whose coefficients $c_j(y)$ are elements of $K[y]$. Each $c_j(y)$ is a combination of finitely many monomials in y, with coefficients in K. If $s \in A$ is a common denominator for those coefficients, then $c_j(x)$ will have coefficients in $A_s[y]$.

We may choose a common denominator s for any finite set of elements of K. Since the generators $\beta_1, ..., \beta_r$ of the algebra B are integral over $k[y]$, we may choose s so that all of those elements are integral over $A_s[y]$. The algebra B_s is generated over A_s by those elements, so it will be an integral extension of A_s. □

localrings

(5.2.6) local rings

A *local ring* R is a noetherian ring that contains just one maximal ideal M. A local ring will have a quotient field $k = R/M$, called the *residue field* of R.

We make a few comments about local rings here, though we will use mainly some special local rings, discrete valuation rings, that will be discussed in the next section.

nonunitideal

5.2.7. Lemma. *A noetherian ring R is a local ring if and only if the set of elements of R that aren't units is an ideal.*

proof. If R is a local ring with maximal ideal M and s is an element of R not in M, then s isn't in any maximal ideal, so it is a unit. And because M isn't the unit ideal, its elements aren't units. Conversely, suppose that the set M of non-units of a ring R is an ideal. Then the unit ideal is the only larger ideal, so M is a maximal ideal. Moreover, if an ideal of R isn't the unit ideal, then its elements aren't units, so it is contained in M. So M is the only maximal ideal. □

Let P be a prime ideal of a noetherian domain A, and let S be the complement of P. The ring of S-fractions is a local ring called the *local ring of A at P*. Thee are various notations for this local ring, one being A_P, though this notation conflicts badly with the notation A_s for $A[s^{-1}]$. The elements of P are the ones that *are not* inverted in the local ring A_P, while in A_s it is the element s that *is* inverted. To make matters even more confusing: If p is a point of an affine variety $X = \operatorname{Spec} A$, the local ring of A at the maximal ideal $\mathfrak{m}_p$ is also often denoted by A_p. Thus if $S = A - \mathfrak{m}_p$, then $A_{\mathfrak{m}_p}$, A_p, and AS^{-1} are three notations for the same local ring.

primelocal **5.2.8. Corollary.** *There is a bijective correspondence between prime ideals of the localization of A at P and prime ideals of A that are contained in P.* □

localizepolyring **5.2.9. Example.** (*localization of the polynomial ring $A = \mathbb{C}[x, y]$*)

Let $\mathfrak{m}$ be the maximal ideal of A at the origin p in $\mathbb{A}^2 = \operatorname{Spec} A$. A polynomial g is in $\mathfrak{m}$ if and only if $g(0, 0) = 0$. So the elements of the local ring $A_\mathfrak{m}$ are fractions of polynomials fg^{-1}, with $g(0, 0) \neq 0$.

The prime ideals of $A_\mathfrak{m}$ are the extensions of the prime ideals of A that are contained in $\mathfrak{m}$. Those prime ideals are: the zero ideal, the ideal $\mathfrak{m}$ itself, and the principal ideals fA generated by irreducible polynomials such that $f(0, 0) = 0$ – the ideals of affine curves C that contain the origin.

Let's denote the set of prime ideals of $A_\mathfrak{m}$ by X_p. When one passes from X to X_p all points except the origin p and all curves that don't contain p disappear. If a curve C contains p, all points except p are gone in X_p, but the origin and what is left of the curve remain. Intuitively, one thinks of X_p as a neighborhood of the origin in the plane. □

figure

The Nakayama Lemma has a version for local rings.

localnakayama **5.2.10. Local Nakayama Lemma.** *Let R be a local ring with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$ and let V be a finite R-module.*

(i) *Let $\overline{V} = V/\mathfrak{m}V$. If $\overline{V} = 0$, then $V = 0$.*

(ii) *Let $S = \{v_1, ..., v_r\}$ be a set of elements of V, whose residues $\overline{v}_1, ..., \overline{v}_r$ span $\overline{V}$. Then S spans V.*

proof. **(i)** If $\overline{V} = 0$, then $V = \mathfrak{m}V$, and there is an element $z \in \mathfrak{m}$ such that $1 - z$ annihilates V. Then $1 - z$ is not in $\mathfrak{m}$, so it is a unit. A unit annihilates V, and therefore $V = 0$.

(ii) Let W be the submodule of V spanned by S. Let the quotient $C = V/W$ is a finite R-module. When we tensor the exact sequence

$$0 \to W \to V \to C \to 0$$

with k, we obtain an exact sequence

$$\overline{W} \to \overline{V} \to \overline{C} \to 0$$

(See Proposition 0.7.6.) We are given that the image of W generates $\overline{V}$. Therefore $\overline{C} = 0$, and by **(i)**, $C = 0$. Therefore $W = V$. □

generatem **5.2.11. Corollary.** *Let R be a local ring with maximal ideal $\mathfrak{m}$ and residue field k. If the residues of a set $S = \{v_1, ..., v_r\}$ of elements of $\mathfrak{m}$ span the k-vector space $\mathfrak{m}/\mathfrak{m}^2$, then S spans $\mathfrak{m}$.* □

dvr 5.3 Valuation Rings

A local domain R with maximal ideal M has *dimension one* if (0) and M are distinct, and are the only prime ideals of R, or if $(0) < M$ is a maximal prime chain in R. In this section, we describe the *normal* local domains of dimension one. They are *discrete valuation rings*.

Let K be a field, and let $K^\times = K - \{0\}$. A *discrete valuation* on K is a surjective map

dval (5.3.1)
$$K^\times \xrightarrow{\text{v}} \mathbb{Z}$$

with these properties:

- $\text{v}(ab) = \text{v}(a) + \text{v}(b)$, i.e., v is a group homomorphism, and
- $\text{v}(a+b) \geq \min\{\text{v}(a), \text{v}(b)\}$, if $a + b \neq 0$.

The word "discrete" refers to the fact that $\mathbb{Z}^+$ is a discrete ordered group. Other valuations exist and they are interesting, but they seem less important, and we won't use them. So to shorten terminology, we will refer to a discrete valuation simply as a *valuation*.

Let k be a positive integer. If v is a valuation and if $\text{v}(\alpha) = k$, then k is the *order of zero of* α, and if $\text{v}(\alpha) = -k$, then k is the *order of pole of* α (with respect to the valuation).

valczero **5.3.2. Lemma.** *Let* v *be a valuation on a field K that contains the complex numbers. Then $\text{v}(c) = 0$ for all nonzero complex numbers c.* □

proof. This is true because $\mathbb{C}$ contains n th roots. The first property of a valuation shows that if $\gamma^r = c$, then $\text{v}(\gamma) = \text{v}(c)/n$. The only integer that is divisible by every integer r is zero. □

The *valuation ring* R associated to a valuation v on a field K consists of the elements of K whose values are non-negative, together with zero:

valnring (5.3.3)
$$R = \{a \in K^\times \mid \text{v}(a) \geq 0\} \cup \{0\}.$$

Valuation rings are often called "discrete valuation rings", but since we have dropped the word discrete from the valuation, we drop it from the ring too.

idealsin-valring **5.3.4. Proposition.** *Let R be the valuation ring of a valuation* v *on a field K.*
(i) *R is a local domain. Its maximal ideal M is the set of elements with positive value:*

$$M = \{a \in K \mid \text{v}(a) > 0\}.$$

This is a principal ideal. It is generated by any element x such that $\text{v}(x) = 1$.
(ii) *The units of R are the elements with value zero. Every nonzero element of K has the form $x^k u$, where u is a unit and k is an integer.*
(iii) *Let N be an R-submodule of K, and assume that $0 < N < K$. Then $N = x^k R$ for some k in $\mathbb{Z}$. The nonzero ideals of R are the powers M^k of M, with $k \geq 0$. Therefore R is noetherian.*
(iv) *If R is a proper subring of a ring R', then $R' = K$. There is no ring R' such that $R < R' < K$.*

proof. **(ii)** Let z be a nonzero element of K and let $\text{v}(z) = k$. Then, with x as in **(i)**, $x^{-k}z$ is a unit in R, so $zR = x^k R$.

(iii) Let N be a nonzero submodule of K and suppose that the values of the elements of N are bounded below. Then if k is the greatest lower bound of those values, $N = x^k R$. If the values of the elements are not bounded below, then N contains $x^k R$ for every k, and $N = K$.
(iv) This follows from **(iii)**. □

valsinCt **5.3.5. Example.** The valuations of the field of rational functions in one variable correspond bijectively to points of the projective line $\mathbb{P}^1$.

proof. Let K denote the field $\mathbb{C}(t)$ of rational functions, and let a be a complex number. To define the valuation v_a that corresponds to the point $t = a$ of $\mathbb{P}^1$, we write a nonzero polynomial as $p = (t-a)^k h$, where $t - a$ doesn't divide h, and we define, $\text{v}_a(p) = k$. We define the value of a nonzero rational function p/q to be $\text{v}_a(p/q) = \text{v}_a(p) - \text{v}_a(q)$. You will be able to check that with this definition, v_a becomes a valuation. The valuation that corresponds to the point at infinity of $\mathbb{P}^1$ is obtained by working with t^{-1} in place of t.

The valuation ring associated to the valuation v_a is the localization of $\mathbb{C}[t]$ at the point $t = a$. Its elements are fractions p/q such that $t - a$ doesn't divide q.

To complete the proof, we show that every valuation v of the field $K = \mathbb{C}(t)$ corresponds to a point of $\mathbb{P}^1$. let R be the valuation ring of a valuation v. If $\mathrm{v}(t) < 0$, then $\mathrm{v}(t^{-1}) > 0$. In that case we replace t by t^{-1}. So we may assume that t is an element of R, and therefore that $\mathbb{C}[t] \subset R$.

The maximal ideal M of R isn't zero. It contains a nonzero element of K, a fraction of polynomials. Since $\mathbb{C}[t] \subset R$, we can clear the denominator in this fraction, while staying in M. So M contains a nonzero polynomial f. Since M is a prime ideal, it contains an irreducible factor of f, of the form $t - a$ for some complex number a. Then $t - a$ is in M. But if $c \neq a$, then $c - a$ isn't in M, and so $t - c$ isn't in M either. It is a unit of R. It follows that R contains the localization R_0 of $\mathbb{C}[t]$ at the point $t = a$, which is a valuation ring. There is no ring properly containing R_0 except K, so $R_0 = R$. □

characterizedvr

5.3.6. Theorem. **(i)** *A local domain R whose maximal ideal M is a nonzero principal ideal is a valuation ring.*
(ii) *The discrete valuation rings are the normal local domains of dimension* 1.

proof. **(i)** Say that M is a nonzero principal ideal, say xR. Let y be a nonzero element of R and let x^k be the largest power of x that divides y (4.1.3). Then $y = ux^k$, where u is in R but not in $M = xR$. Since R is a local ring, u is a unit. Then any nonzero element z of the fraction field K of R has the form $z = vx^r$ where r is a positive or negative integer and v is a unit. This is seen by writing the numerator and denominator of a fraction in such a form and dividing. The valuation whose valuation ring is R is defined by $\mathrm{v}(z) = r$, where r is as above. If $z_i = v_i x^{r_i}$, $i = 1, 2$, where v_i is a unit and $r_1 \leq r_2$, then $z_1 + z_2 = \alpha x^{r_1}$, where $\alpha = v_1 + v_2 x^{r_2 - r_1}$ is an element of R. Therefore $\mathrm{v}(z_1 + z_2) \geq r_1 = \min\{\mathrm{v}(z_1), \mathrm{v}(z_2)\}$. The requirements for a valuation are satisfied.

(ii) The normalization R' of a discrete valuation ring R is a finite R-module contained in the fraction field K. Since K isn't a finite R-module, Proposition 5.3.4 **(iii)** shows that $R = R'$.

Conversely, let R be a normal local domain of dimension 1. We show that R is a valuation ring by showing that the maximal ideal of R is a principal ideal. Let α be a nonzero element of M. Because R has dimension 1, M is the only prime ideal that contains α, so M is the radical of the principal ideal αR, and $M^r \subset \alpha R$ for large r. Let r be the smallest such integer. Then $r > 0$. If $r = 1$, then $M = \alpha R$ so M is a principal ideal. If $r > 1$, there is an element β in M^{r-1} such that $\beta \notin \alpha R$, but $\beta M \subset \alpha R$. Let $\gamma = \beta/\alpha$. Then $\gamma \notin R$, but $\gamma M \subset R$. Since M is an ideal, multiplication by an element of R carries γM to itself. So γM is an ideal too. Since R is a local ring with maximal ideal M, either $\gamma M \subset M$ or $\gamma M = R$. If $\gamma M \subset M$, the lemma elow shows that γ is integral over R. This is impossible because R is normal and $\gamma \notin R$. Therefore $\gamma M = R$. Then $M = \gamma^{-1} R$. This implies that γ^{-1} is in R, and that M is a principal ideal. □

betaintegral

5.3.7. Lemma. *Let I be a nonzero ideal of a noetherian domain A, and let B be a domain that contains A. An element γ of B such that $\gamma I \subset I$ is integral over A.*

proof. This is the Nakayama Lemma again. Because A is noetherian, I is finitely generated. Let $v = (v_1, ..., v_n)^t$ be a vector whose entries generate I. The hypothesis $\gamma I \subset I$ allows us to write $\gamma v_i = \sum p_{ij} v_j$ with p_{ij} in A, or in matrix notation, $\gamma v = Pv$. Let $p(t)$ be the characteristic polynomial of P. Then $p(\gamma)v = 0$. Since $I \neq 0$, at least one v_i is nonzero. Therefore, since A is a domain, $p(\gamma) = 0$. The characteristic polynomial is a monic polynomial with coefficients in A, so γ is integral over A. □

smaffcurve

5.4 Smooth Affine Curves

A *curve* is a variety of dimension 1. Its proper closed subsets are the finite sets.

Let $X = \operatorname{Spec} A$ be an affine curve. A rational function is regular on X if and only if it is regular at every point p, which means that it is in every every local ring A_p. But we also know that α is regular if and only if it is an element of A (Proposition 3.4.3). Therefore the coordinate ring A of an affine curve $X = \operatorname{Spec} A$ is the intersection of its localizations:

intersectcodimoneagain

(5.4.1)
$$A = \bigcap A_p \quad (\text{in } K)$$

In fact, this is true for any affine variety. consequence is that a domain A is normal if and only if all of its localizations A_p are normal. (This follows from Lemma 4.3.3 **(ii)**).

A point p of a curve X is a *smooth point* if the local ring at p is a valuation ring, and a curve is *smooth* if all of its points are smooth. Thus an affine curve X is smooth if and only if its coordinate algebra is a normal domain (Theorem 5.3.6).

If a curve X is smooth at p, we denote the the corresponding valuation by v_p. The *zeros* Z and the *poles* P of a rational function α on a smooth curve X are defined as the points p at which α has a zero or a pole, with respect to the valuation v_p.

pointsvalns **5.4.2. Proposition.** *Let $X = \operatorname{Spec} A$ be a smooth affine curve. The localizations A_p of A at the points p of X are the valuation rings of the fraction field K that contain A.*

proof. First, the localization A_p at a point p is a valuation ring that contains A (Theorem 5.3.6). Let R be a valuation ring of K that contains A, let v be the associated valuation, and let M be the maximal ideal of R. The intersection $M \cap A$ is a prime ideal of A. Since A has dimension 1, the zero ideal is the only prime ideal of A other than the maximal ideals. To verify that $M \cap A$ isn't the zero ideal, we choose a nonzero element $\alpha \in M$, and write it as a fraction a/b, with a and b in A. Then $\mathrm{v}(a) \geq \mathrm{v}(\alpha) > 0$, so $a = b\alpha$ is a nonzero element of $M \cap A$.

Since $M \cap A$ isn't zero, it is the maximal ideal $\mathfrak{m}_p$ of A corresponding to a point p of X. The elements of A not in $\mathfrak{m}_p$ aren't in M either, and they are invertible in R. Therefore the local ring A_p, at p, which is a valuation ring, is contained in R. So $A = R$ (5.3.4 **(iii)**). □

pointsofcurve **5.4.3. Corollary.** *Let X be a smooth curve, not necessarily affine, with function field K. Morphisms $X \to \mathbb{P}^n$ correspond bijectively to points of $\mathbb{P}^n$ with values in K.*

proof. Let $(\alpha_0, ..., \alpha_n)$ be a point of $\mathbb{P}^n$ with values in K. Proposition **??** tells us that α determines a morphism $X \to \mathbb{P}^n$ if and only if, for every point p of X, there is an index i such that the functions α_j/α_i are regular at p for every j. The functions α_j/α_i will be regular at p when i is chosen so that the order of zero $\mathrm{v}_p(\alpha_i)$ of α_i at p is minimal. □

This Corollary isn't true in dimension greater than one. If X is the affine plane $\operatorname{Spec} \mathbb{C}[x, y]$, its function field K is the field $\mathbb{C}(x, y)$ of rational functions. The pair of functions x, y defines a point of $\mathbb{P}^1$ with values in K, but not a morphism $X \to \mathbb{P}^1$. There is no way to extend the map to the origin.

onezero **5.4.4. Lemma.** *Let X be a smooth affine curve with coordinate algebra A and function field K, and let p be a point of X. There exists an element α in K with pole of order 1 at p and no other pole.*

If the maximal ideal $\mathfrak{m}_p$ of A at p is a principal ideal, a generator t will have p as its only zero. Then t^{-1} will have p as its only pole, and it will have no zeros. If $\mathfrak{m}_p$ isn't a principal ideal, the element we are looking for will have some zeros as well as its single pole.

proof of the lemma. Let R denote the local ring A_p at p, and let t be an element of A that generates the maximal ideal of R. Then t will have a zero of order 1 at p, and because X has dimension one, it will have finitely many other zeros, say $q_1, ..., q_r$. There is an element z of A that is zero at $q_1, ..., q_r$ but not zero at p. Then for large n, $z^n t^{-1}$ will be an element of K with a pole of order 1 at p, and no other pole. □

truncatecurve **5.4.5. Proposition.** *Let $X = \operatorname{Spec} A$ be a smooth affine curve, and let $\mathfrak{m}$ be the maximal ideal of A at a point p of X. If I is an ideal whose radical is $\mathfrak{m}$, then I is a power $\mathfrak{m}^k$ of $\mathfrak{m}$.*

proof. Let v be the valuation corresponding to the point p, and let R be the associated valuation ring, the local ring of A at p. The nonzero ideals of R are powers of its maximal ideal M.

The maximal ideal $\mathfrak{m}$ consists of the elements a of A with value $\mathrm{v}(a) \geq 1$. Therefore $\mathfrak{m}^r$ contains elements that have value r, and all nonzero elements of $\mathfrak{m}^r$ have value at least r. Let k be the minimal value $\mathrm{v}(x)$ among the nonzero elements x of I. Every nonzero element of I has value at least k. We will show that I is the set of all elements y of A with $\mathrm{v}(y) \geq k$. Since we can apply the same reasoning to $\mathfrak{m}^k$, it will follow that $I = \mathfrak{m}^k$.

Let y be a nonzero element of A with $\mathrm{v}(y) \geq k$. Then since $\mathrm{v}(uy) \geq \mathrm{v}(x)$, x divides y in R. So we may write y in the form $y = s^{-1}ax$, where s, a are in A, and $s \notin \mathfrak{m}$. The element s will vanish at a finite set of points $q_1, ..., q_r$ distinct from p.

We choose an element s' of A that vanishes at p but not at any of the points $q_1, ..., q_r$, and we look at the localization $A_{s'}$. The extended ideal $\mathfrak{m}A_{s'}$ is the unit ideal. Since the radical of I is $\mathfrak{m}$, the localized ideal $I_{s'}$

is the unit ideal too. Therefore y is in I_t. We may write $y = s'^{-n}b$ for some $b \in I$. Since we can replace s' by a power, we may assume that $y = s'^{-1}b$. We now have the two equations

$$sy = ax \quad \text{and} \quad s'y = b$$

among elements of A. By our choice, s' and s have no common zeros in $X = \operatorname{Spec} A$. They generate the unit ideal of A. Writing $us + u's' = 1$ with u, u' in A, we have $y = (us + u's')y = uax + u'b$. The right side of this equation is in I, so y is in I. □

openaffine **5.4.6. Corollary.** *Every nonempty open subvariety X' of a smooth affine curve X is a smooth affine curve.*

proof. A nonempty open subset of a curve is the complement of a finite set, so it will be enought to consider the case of the open set X' obtained by deleting a single point p of X. Lemma 5.4.4 tells us that there is an element α in K with a pole at p and no other pole. Let A_1 denote the finite type domain $A[\alpha]$. We show that $X_1 = \operatorname{Spec} A_1$ is isomorphic to X'.

The inclusion $A \subset A_1$ gives us a morphism $X_1 \xrightarrow{u} X$. If q is a point of X different from p, then α is an element of the local ring A_q. Therefore $A_1 \subset A_q$, and so there is a point q_1 of X_1 that maps to q. Since A_q is a valuation ring, $A_q = A_{1q_1}$ (5.3.4 **(iii)**. So q_1 is the only point of X_1 that lies over q. One the other hand, since $\alpha \notin A_p$ but $\alpha \in A_1$, there is no point of X_1 lying over p. So the map u sends X_1 bijectively to $X' = X - \{p\}$. The map is a homeomorphism simply because the proper closed sets in X_1 and in X' are the finite sets. To show that the inverse map $X' \xrightarrow{v} X_1$ is an isomorphism, we must show that if a rational function β is regular at a point q_1 of X_1, then it is regular at $q = v(q_1) = u^{-1}(q_1)$. This is true because the local rings are equal. □

jacob **(5.4.7) the Jacobian criterion**

smoothcurvedef **5.4.8. Proposition.** *Let $X = \operatorname{Spec} A$ be an affine curve with coordinate algebra $A = \mathbb{C}[x_1, ..., x_n]/(f_1, ..., f_k)$. A point p of X is smooth if and only if the Jacobian matrix $J = \frac{\partial f_i}{\partial x_j}$ has rank $n - 1$ at p.*

We leave the proof as an exercise. □

This Jacobian criterion generalizes to higher dimension. An affine variety X of dimension d whose coordinate algebra is presented as $A = \mathbb{C}[x_1, ..., x_n]/(f_1, ..., f_k)$ is *smooth* at a point p if and only if the Jacobian matrix $J = \frac{\partial f_i}{\partial x_j}$, evaluated at p, has rank $n - d$. However, to apply this criterion, one needs to know the dimension of X, and the dimension may not be easy to detrmine.

twistcubic **5.4.9. Example.** The *twisted cubic* X in $\mathbb{P}^3$ is the curve whose points are $(1, t, t^2, t^3)$ for $t \in \mathbb{C}$ together with the point $(0, 0, 0, 1)$. It is defined by the three homogeneous equations

twistcubicequations (5.4.10)
$$x_0x_3 = x_1x_2, \quad x_1^2 = x_0x_2, \quad x_2^2 = x_1x_3$$

The zero locus of the first two equations is the union of the twisted cubic and the line $L : x_0 = x_1 = 0$, and the last equation eliminates all points of the line except $(0, 0, 0, 1)$. The rank of the Jacobian matrix is 2 at all points of X, so X is a smooth projective curve. □

5.5 Nodes and Cusps II

nodecuspatwo We describe nodes and cusps of curves here. Nodes and cusps of plane curves were defined in Chapter **??**.

Let p be a singular point of a curve X. For simplicity, let's assume that X is affine and that p is its only singular point. We can achieve this by localizing. Let $k = k(p)$ be the residue field at p, and let $\widetilde{X} = \operatorname{Spec} \widetilde{A}$ be the normalization of X.

defnode **5.5.1. Definition.** The point p is a node or a cusp if and only if the A-module $\epsilon = \widetilde{A}/A$ has dimension one as complex vector space, i.e., if and only if ϵ is isomorphic, as module, to the residue field $k = k(p)$. If ϵ has dimension one, and if there are two points of $\widetilde{X}$ lying over p, then p is called a *node*, while if there is just one point lying over p, then p is called a *cusp*.

twopoints **5.5.2. Lemma.** *If ϵ has dimension one, the quotient algebra $\widetilde{A}/\mathfrak{m}\widetilde{A}$ has dimension two. Therefore there are at most two points of $\widetilde{X}$ that lie over p.*

proof. It is convenient to form a diagram in which $\mathfrak{m}$ denotes the maximal ideal of A at p and $k = A/\mathfrak{m}$ denotes the residue field at p:

AoA (5.5.3)

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathfrak{m} & \longrightarrow & \mathfrak{m}\widetilde{A} & \longrightarrow & \mathfrak{m}\epsilon & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & \widetilde{A} & \longrightarrow & \epsilon & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & k & \xrightarrow{i} & \widetilde{A}/\mathfrak{m}\widetilde{A} & \longrightarrow & \epsilon/\mathfrak{m}\epsilon & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

The middle row and all three columns are exact. Since ϵ is isomorphic to k, $\mathfrak{m}\epsilon = 0$ and therefore $\epsilon \approx \epsilon/\mathfrak{m}\epsilon$. The Snake Lemma, applied to the first two columns, shows that $\mathfrak{m} \approx \mathfrak{m}\widetilde{A}$, and that all rows are exact. Then the bottom row shows that $\widetilde{A} \otimes_A k$ has dimension 2. □

describenode **5.5.4. Proposition.** **(i)** *Suppose that p is a node, and let q_1 and q_2 be the points of $\widetilde{X}$ over p. Then A is the subalgebra of $\widetilde{A}$ of elements α such that $\alpha(q_1) = \alpha(q_2)$.*
(ii) *Suppose that p is a cusp. Let $\widetilde{\mathfrak{m}}$ be the maxmial ideal of $\widetilde{A}$ at the point q of $\widetilde{X}$ over p. Then A is the subalgebra $k + \widetilde{\mathfrak{m}}^2$ of $\widetilde{A}$.*

*proof.***(i)** Let A' be the subalgebra of $\widetilde{A}$ of elements α such that $\alpha(q_1) = \alpha(q_2)$. It is obvious that $A \subset A'$, and that $A' < \widetilde{A}$. Since ϵ has dimension one, $A = A'$.

(ii) The only maximal ideal of $\widetilde{A}$ that contains $\mathfrak{m}$ is the maximal ideal $\widetilde{\mathfrak{m}}$ at the single point $\widetilde{p}$ that lies over p. Therefore the radical of the ideal $\mathfrak{m}\widetilde{A}$ is $\widetilde{\mathfrak{m}}$, and $\mathfrak{m}\widetilde{A}$ is a power of $\widetilde{\mathfrak{m}}$ (Proposition 5.4.5). Since $\widetilde{A}/\mathfrak{m}\widetilde{A}$ has dimension 2, $\mathfrak{m}\widetilde{A} = \widetilde{\mathfrak{m}}^2$. □

construct

5.6 Constructible Sets

In this section, X denotes a noetherian topological space. Every strictly decreasing chain of closed subsets of X is finite, and every closed subset is a union of finitely many irreducible closed sets.

The intersection $L = C \cap U$ of a closed subset C and an open subset U of X is a *locally closed set*. For instance, closed subsets and open subsets are locally closed. A subset is *constructible* if it is the union of finitely many locally closed sets.

In this section we use the following notation: L is locally closed, C is closed, and U is open.

constrincurve **5.6.1. Example.** A subset S of a curve X is constructible if and only if it is either a finite set or the complement of a finite set. Thus S is constructible if and only if it is either closed or open, in which case it is locally closed. □

The proofs of the next two theorems are elementary topology, but they are confusing enough to require care.

deflocclosed **5.6.2. Theorem.** *The set $\mathbb{S}$ of constructible subsets of a noetherian topological space X is the smallest set of subsets that contains the open sets and is closed under the three operations of finite union, finite intersection, and complementation.*

proof. Let $\mathbb{S}_1$ denote the set of subsets that is obtained from the open sets by the three operations mentioned in the statement. Open sets are constructible, and with those operations, one can produce any constructible set from the open sets. So $\mathbb{S} \subset \mathbb{S}_1$.

To show that $\mathbb{S} = \mathbb{S}_1$, we show that the constructible sets are closed under the three operations. It is obvious that a finite union of constructible sets is constructible. The intersection of two locally closed sets $L_1 = C_1 \cap U_1$ and $L_2 = C_2 \cap U_2$ is locally closed because $L_1 \cap L_2 = (C_1 \cap C_2) \cap (U_1 \cap U_2)$. If $S = L_1 \cup \cdots \cup L_k$ and $S' = L'_1 \cup \cdots \cup L'_r$ are constructible sets, the intersection $S \cap S'$ is equal to the union $\bigcup (L_i \cap L'_j)$, so it is constructible.

To show that the complement S^C of a constructible set S is constructible, it suffices to show that the complement of a locally closed set is constructible. For, if $S = L_1 \cup \cdots \cup L_k$, then $S^C = L_1^C \cap \cdots \cap L_k^C$, and we know now that intersections of constructible sets are constructible. Let L be the locally closed set $C \cap U$, and let $V = C^C$ and $Y = U^C$ be the complements of C and U, respectively. Then V is open and Y is closed. The complement L^C of L is the union $V \cup Y$ of constructible sets, so it is constructible. □

containsopen **5.6.3. Theorem.** **(i)** *Every constructible set S is a union $L_1 \cup \cdots \cup L_k$ of locally closed sets $L_i = C_i \cap U_i$, in which the closed sets C_i are irreducible and distinct.*

(ii) *Let $\overline{S}$ be the closure of a nonempty constructible set S. There is a nonempty locally closed subset L' of S that is an open subset of $\overline{S}$.*

proof. **(i)** Suppose that a locally closed set L has the form $C \cap U$, and let $C = C^1 \cup \cdots \cup C^r$ be the decomposition of C into irreducible components. Then $L = (C^1 \cap U) \cup \cdots \cup (C^r \cap U)$. So if a constructible set S is written as a union $L_1 \cup \cdots \cup L_k$, we can replace each L_i by a union of locally closed sets of the form $C \cap U$, where C is irreducible. Next, say that $C_1 = C_2$. Then $L_1 \cup L_2 = (C_1 \cap U_1) \cup (C_1 \cap U_2) = C_1 \cap (U_1 \cap U_2)$ is locally closed. So we can find an expression in which the irreducible closed sets are C_i distinct.

(ii) Say that $S = L_1 \cup \cdots \cup L_k$ and that $L_i = C_i \cap U_i$, where the sets C_i are irreducible and distinct. The closure of S is the union $\overline{S} = C_1 \cup \cdots \cup C_k$. We choose an index i, say $i = 1$, so that C_1 isn't contained in C_i for $i > 1$, and we let $Z = C_2 \cup \cdots \cup C_k$ be the union of the remaining sets C_i. Let V be the open complement of Z in X, and let $W = \overline{S} \cap V$. Every point of W lies in C_1, so it is also true that $W = C_1 \cap V$. Therefore W is an open subset of C_1 and of $\overline{S}$. It is nonempty because it contains the points of C_1 that aren't in Z. Let

$$L' = L_1 \cap W = (C_1 \cap U_1) \cap (C_1 \cap V) = (C_1 \cap C_2) \cap (U_1 \cap U_2)$$

This is an intersection of nonempty open subsets of C_1, and is therefore a nonempty locally closed set whose closure is C_1. Since L_1 is open in C_1, L' is open in W, and therefore open in C_1 and in $\overline{S}$. Since $L' \subset L_1 \subset S$, L' is the required subset. □

sinx **5.6.4. Proposition.** **(i)** *Let X' be an open or a closed subvariety of a variety X. A subset S of X' is a constructible subset of X' if and only if it is a constructible subset of X.*

(ii) *Let S be a subset of a variety X, let Y be a closed subset of X, and let V be the open complement of Y in X. Then S is constructible if and only if $S \cap Y$ and $S \cap V$ are constructible.*

proof. **(i)** Let $L' = C' \cap U'$ be a locally closed subset of X', with C' closed and U' open in X'. If X' is open in X, then U' is also open in X, and if C denotes the closure of C' in X, $L = C \cap U'$. So L is locally closed in X. Conversely, if $L = C \cap U$ is locally closed in X, then $L = C' \cap U$, where $C' = C \cap X'$ is closed in X'. If X' is closed in X, and if V is the complement of X' in X, then C' is closed in X, and $L' = C' \cap (U' \cup V)$. □

The next theorem illustrates a general principle: Sets that arise in algebraic geometry are often constructible.

imageconstr **5.6.5. Theorem.** *Let $Y \xrightarrow{f} X$ be a morphism of varieties.*

(i) *The inverse image of a constructible subset of X is a constructible subset of Y.*

(ii) *The image of a constructible subset of Y is a constructible subset of X.*

proof. Part **(i)** follows directly from the fact that f is a continuous map. The proof of **(i)** is brutal. One hammers away until there is nothing left to do.

Let S be a constructible subset of Y.

Step 1: Suppose that Y is the union of finitely many subvarieties, which may be open or closed, and let $S_j = S \cap Y_j$. Then S_j are constructible and their union is S. It suffices to show that the image of each S_j is constructible. Similarly, suppose that X is the union of finitely many open or closed subvarieties X_i. Let $Y_i = f^{-1}X_i$ and let $S_i = S \cap Y_i$. It suffices to show that the image of each S_i is constructible. Moreover, Proposition 5.6.4 tells us that the image of S_i is constructible in X_i if and only if it is constructible in X.

Step 2: Noetherian induction on Y and on X allows us to assume that the image $f(S)$ is constructible if S is contained in a proper closed subset of Y, or if $f(S)$ is contained in a proper closed subset of X. Therefore we may assume that Y is the closure of S and that X is the closure of $f(S)$.

When we decompose X into a proper closed subvariety X_1 and an open subvariety X_2, and we decompose Y by the inverse images $Y_i = f^{-1}X_i$, noetherian induction applies to the map $Y_1 \to X$. Similarly, when we decompose Y in to a proper closed subvariety Y_1 and an open subvariety Y_2, noetherian induction applies to the map $Y_1 \to X$. In either case, we may ignore Y_1. This means that we may replace X by any nonempty open subvariety X' and Y by any nonempty open subvariety of $f^{-1}X'$. We can do this finitely many times.

Step 3: Since Y is the closure of S, Theorem 5.6.3 **(ii)** tells us that S contains a nonempty open subset of Y. We may replace Y by that subset. So it suffices to show that the image of Y itelf is constructible. And as Step 2 shows, we may assume that the closure of $f(Y)$ is X.

W may still replace X and Y by nonempty open sets, so we may assume that X and Y are affine, say $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, and that the morphism f corresponds to the algebra homomorphism $A \xrightarrow{\varphi} B$. If the kernel of φ were a nonzero (prime) ideal P, the image of Y would be contained in a proper closed subset of X. We have taken care of that case. So φ is injective.

Corollary 5.2.5 tells us that, for suitable nonzero s in A, B_s is a finite module over a polynomial subring $A_s[y]$. Then both of the maps $Y_s \to \operatorname{Spec} A_s[y]$ and $\operatorname{Spec} A_s[y] \to X_s$ are surjective, so Y_s maps surjectively to X_s. When we replace X and Y by X_s and Y_s, the map becomes surjective, and we are done. □

usingcurves

5.7 Closed Sets

Limits of sequences are often used to analyze subsets of a topological space. A metric space Y is closed in the classical topology if, whenever a sequence of points in Y has a limit in $\mathbb{R}^n$, the limit is in Y. In algebraic geometry one uses morphisms from algebraic curves to Y as a substitute. We use the following notation to state the analogue:

Cwithpoint (5.7.1) *C is a smooth affine curve, and $C' = C - q$ is the complement of a point q of C.*

The (Zariski) closure of C' will be C, and we think of q as a limit point. Theorem 5.7.3, which is below, asserts that a constructible subset Y of a variety X is closed if it contains all such limit points. It is based on the next theorem, which states that there are enough curves to do the job.

enoughcurves **5.7.2. Theorem.** *(there are enough curves) Let Y be a constructible subset of a variety X, and let p be a point of its closure $\overline{Y}$. There exist a morphism $C \xrightarrow{f} X$ from a smooth curve to X and a point q of C such that $f(q) = p$ and $f(C') \subset Y$.*

proof. The method is to use Krull's Theorem to slice Y down to dimension 1.

If $X = p$, we may take for f the constant morphism from any curve C to p. So we may assume that X has positive dimension d. Next, we may replace X by any affine open subset that contains p, and Y and $\overline{Y}$ by their intersections with that open subset. So we may assume X affine, say $X = \operatorname{Spec} A$.

Since Y is constructible, it is union $L_1 \cup \cdots \cup L_k$ of locally closed sets $L_i = C_i \cap U_i$, where the closed sets C_i are irreducible. The closure of Y is $\overline{Y} = C_1 \cup \cdots \cup C_k$. Since p is in $\overline{Y}$, it is in one of the closed sets C_i. We may replace Y by L_i and X by C_i, so we may assume that Y is a nonempty open subset of X.

Suppose that the dimension d of X is at least two. Let $W = X - Y$ be the complement of Y in X. The components of W have dimension at most $d-1$. We choose a suitable element $\alpha \in A$ such that $\alpha(p) = 0$: We require that α isn't identically zero on any component of W, except for p, if p happens to be a component. Krull's Theorem tells us that every component of the zero locus $V_X(\alpha)$ of α has dimension $d-1$, and at least one of those components contains p. Let $\overline{V}$ be such a component. Since α isn't identically zero on any component of W different from p, and since p has codimension at least two in X, $\overline{V} \not\subset W$. So $V = \overline{V} \cap Y$ is a nonempty open, and therefore dense, subset of $\overline{V}$, and p is a point of its closure. We replace X by $\overline{V}$ and Y by V, which reduces us to the case that the dimension of X is $d-1$.

Thus it suffices to treat the case that X has dimension 1. Then X will be an affine curve that contains p. Its normalization will be a smooth affine curve C_1 that comes with a surjective morphism to X. Finitely many points of C_1 will map to p. We delete all but one of those points to obtain the required affine curve C. □

closedcrittwo **5.7.3. Theorem** *(curve criterion for a closed set) Let Y be a constructible subset Y of a variety X. The following conditions are equivalent:*
(a) *Y is a closed subset of X.*
(b) *Let $C \xrightarrow{f} X$ be a morphism from a smooth affine curve to X. The inverse image $f^{-1}Y$ is closed in C.*
(c) *Let q be a point of a smooth affine curve C, let $C' = C - \{q\}$, and let $C \xrightarrow{f} X$ be a morphism. If $f(C') \subset Y$, then $f(C) \subset Y$.*

The hypothesis that Y be constructible is necessary. The set Y of points of $\mathbb{A}^n$ with integer coordinates isn't constructible, but it satisfies the curve criterion. Any morphism $C' \to X$ whose image is in Y will map C' to a single point, and therefore it will extend to C.

proof. The implications **(a)** ⇒ **(b)** ⇒ **(c)** are obvious. We prove the contrapositive of the remaining implication **(c)** ⇒ **(a)**. Suppose that Y isn't closed. We choose a point p of the closure $\overline{Y}$ that isn't in Y, and we apply Theorem 5.7.2. There exists a morphism $C \xrightarrow{f} X$ from a smooth curve to X and a point q of C such that $f(q) = p$ and $f(C') \subset Y$. This morphism shows that **(c)** doesn't hold either. □

5.8 Fibred Products

fibprod

First, fibred products of sets. If $X \xrightarrow{f} Z$ and $Y \xrightarrow{g} Z$ are maps of sets, the *fibred product* $X \times_Z Y$ is the subset of the product $X \times Y$ consisting of pairs of points x, y such that $f(x) = g(y)$. The fibred product fits into a diagram

fproddiagr (5.8.1)

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{\pi_Y} & Y \\ {\scriptstyle \pi_X}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Z \end{array}$$

where π_X and π_Y are the projections. The reason for the term "fibred product" is that each fibre of $X \times_Z Y$ over a point of X maps bijectively to a fibre of Y over a point of Z.

Many important subsets of a product can be realized as fibred products. If $p \to Z$ is the inclusion of a point into Z, then $p \times_Z Y$ is the fibre of Y over p. The product $X \times_X X$ is the diagonal in $X \times X$.

Now, fibred products of varieties Since we are working with varieties and not with general schemes, we have a small problem: A fibred product of varieties will always be a scheme, but it needn't be a variety.

example-fibred-product **5.8.2. Example.** Let X, Y and Z be affine lines, let $X \xrightarrow{f} Z$ be the map $z = x^2$, and let g be the map $z = y^2$. The fibred product $X \times_Z Y$ is the closed subset of the affine x,y-plane consisting of the diagonal $x = y$ and the antidiagonal $x = -y$.

Rather than discussing schemes, we show that the fibred product of varieties is a (Zariski) *closed subset* of the product $X \times Y$. This will be enough for our purposes.

fibprodclosed **5.8.3. Proposition.** *Let $X \xrightarrow{f} Z$ and $Y \xrightarrow{g} Z$ be morphisms of varieties. The fibred product $X \times_Z Y$ is a closed subset of the product variety $X \times Y$.*

proof. Step 1. The graph Γ_f of a morphism $X \xrightarrow{f} Z$ is a closed subvariety of $X \times Z$ that is isomorphic to X: The graph can be represented as a fibred product by the diagram

$$\begin{array}{ccc} \Gamma_f & \longrightarrow & X\times Z \\ \big\downarrow & & \big\downarrow{\scriptstyle f\times id} \\ Z_\Delta & \xrightarrow{\Delta} & Z\times Z \end{array}$$

where Z_Δ is the diagonal, a closed subset of $Z{\times}Z$. The map $F{\times}id$ is a morphism, and Γ_f is the inverse image in $X\times Z$ of the closed subvariety Z_Δ of $X\times X$, so it is a closed subset of $X\times Z$.

The projection of Γ_f to X is bijective. It is continuous because the projection $X{\times}Z \xrightarrow{\pi} X$ is a morphism. Its inverse is obtained using the mapping property of product varieties (Proposition 3.1.16), which gives us a morphism $(id,f) : X{\longrightarrow}X\times Z$, whose image is Γ_f. Therefore X and Γ_f are homeomorphic. This shows that Γ_f is an irreducible closed set, and therefore a closed subvariety, of $X\times Z$. The maps $X \to \Gamma_f$ and $\Gamma_f \to X$ we have described are inverse morphisms, so Γ_f is isomorphic to X.

Step 2. Let u and v be two morphisms from a variety X to another variety z: $X\rightrightarrows Z$. The set W consisting of points x in X such that $u(x) = v(x)$ is closed in X: Let $W' = \Gamma_u \cap \Gamma_v$ in $X{\times}Z$. This is the intersection of Γ_u with the closed set Γ_v, so W' is closed in Γ_u. The isomorphism $\Gamma_u \to X$ carries W' to W, so W is closed in X.

Step 3: Completion of the proof. With reference to diagram 5.8.1, $X\times_Z Y$ is the subset of $X\times Y$ of points at which the maps $f\pi_X$ and $g\pi_Y$ to Z are equal. □

For reference in the next section, we derive a corollary of Theorem 5.7.2.

liftcurve **5.8.4. Corollary.** *(lifting of curves) Let $W \xrightarrow{u} Z$ and $C \xrightarrow{f} Z$ be morphisms of varieties, where C is a smooth affine curve. If the image $f(C)$ is contained in the image $u(W)$, there is a smooth affine curve D that fits into a diagram of morphisms*

$$\begin{array}{ccc} D & \xrightarrow{f'} & W \\ {\scriptstyle g}\big\downarrow & & {\scriptstyle u}\big\downarrow \\ C & \xrightarrow{f} & Z \end{array}$$

such that g isn't a constant map – it isn't the map from D to a single point.

In this corollary, we can't require that the map g be surjective. Its image will be a nonempty open subset of C.

proof. We form the fibred product $C\times_Z W$. Since $f(C)$ is contained in $u(W)$, the projection from $C\times_Z W$ to C is surjective. At least one component of $C\times_Z W$ will map to an open subset of C. We choose such a component, call it W', and we project W' to its image, a nonempty open subset C' of C. Since the map g we are looking for isn't required to be surjective, replacing C by C' is permissible, and we do that. Then we are looking for a smooth curve D and morphisms to complete the diagram below:

$$\begin{array}{ccccc} D & \longrightarrow & W' & \longrightarrow & W \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle u'} & & \big\downarrow{\scriptstyle u} \\ C & \xrightarrow{=} & C & \longrightarrow & Z \end{array}$$

We replace the map $W \xrightarrow{u} Z$ by $W' \xrightarrow{u'} C$. This reduces us to the case that f is the identity map $C \to C$. When we drop the primes from W' and u', the problem becomes this: Let $W \xrightarrow{u} C$ be a surjective morphism to a smooth affine curve C. There exists a smooth affine curve D and a morphism $D \xrightarrow{f'} W$ such that the composed morphism $D \xrightarrow{uf'} C$ isn't a constant map.

Let p_1 be an arbitrary point of W, let p be its image in C, and let F be the fibre of the map u over p. Theorem 5.7.2 shows that there is a map $D \xrightarrow{f'} W$ from a smooth affine curve D to W and a point q of D such that $f'(q) = p_1$, and that the image of $D' = D - \{q\}$ is contained in the complement of F. This is the required map. □

proper 5.9 Projective Varieties are Proper

An important property of projective space is that, with its classical topology, it is *compact*, which means that it has these two properties: It is a *Hausdorff space*: Distinct points p, q of X have disjoint open neighborhoods, and it is *quasicompact*: If family $\{X_i\}$ of open sets covers X, then a finite subfamily covers X.

The next theorem reviews two important properties of compact spaces.

heineborel **5.9.1. Theorem.** **(i)** *(Heine-Borel Theorem) A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.*

(ii) *Let $X \xrightarrow{f} Y$ be a continuous map of topological spaces. Suppose that X is a compact space and that Y is a Hausdorff space. The image $f(X)$ is a closed subset of Y, and with the topology induced from Y, the image is compact.* □

Let's use this theorem to verify that $\mathbb{P}^n$ is compact. The five-dimensional sphere $\mathbb{S}$ of unit length vectors in $\mathbb{C}^{n+1}$ is bounded, and because it is the zero locus of the equation $\overline{x}_0 x_0 + \cdots + \overline{x}_n x_n = 1$, it is closed. So it is compact. The map $\mathbb{S} \to \mathbb{P}^2$ that sends a vector $(x_0, ..., x_n)$ to the point of the projective plane with that coordinate vector is continuous and surjective. So $\mathbb{P}^n$ is compact.

We saw in Section 2.7 that, in the Zariski topology, every variety is a notherian topological space. Consequently, it is quasicompact. But a variety of dimension > 0 isn't compact because it isn't Hausdorff. We show that projective varieties have a property closely related to compactness: They are *proper*.

Before defining proper varieties, we explain the analogous property of compact spaces.

propercompact **5.9.2. Proposition.** *Let W be a closed subset of a product $Z \times X$, where Z is a Hausdorff space and X is a compact space. The image Y of W via projection to Z is a closed subset of Z.*

proof. Let y_i be a sequence of points of Y that has a limit $\overline{y}$ in Z. We show that $\overline{y}$ is a point of Y. For each i, we choose a point w_i of W that lies over y_i. The point w_i is a pair (y_i, x_i), x_i being a point of X. Since X is compact, there is a subsequence of x_i that has a limit $\overline{x}$ in X. Passing to subsequences, we may suppose that x_i has limit $\overline{x}$. Then w_i has the limit $(\overline{y}, \overline{x})$. Since W is closed, $(\overline{y}, \overline{x})$ is in W, and therefore $\overline{y}$ is in Y. □

The property of this proposition defines proper varieties.

defproper **5.9.3. Definition.** A variety X is *proper* if for every variety Z and every closed subset W of the product $Z \times X$, the image Y of W via projection to Z is closed in Z.

Because the image of an irreducible subset is irreducible. the image of a closed subvariety Z of $Y \times X$ will be a closed subvariety of Y, if X is proper.

pnproper **5.9.4. Theorem.** *Projective varieties are proper.*

This theorem is the most important application of the use of curves to characterize closed sets. Before proving it, we give some examples which show how it is used.

properex **5.9.5. Example.** (*singular curves*) We assemble the plane curves of a given degree d into a variety. The number of distinct monomials $x_0^i x_1^j x_2^k$ of degree $d\ = i+j+k$ is the binomial coefficient $\binom{d+2}{2}$. We order the monomials arbitrarily, labeling them as $m_0, ..., m_r$, $r = \binom{d+2}{2} - 1$. A homogeneous polynomial of degree d will be a combination of monomials with complex coefficients $z_0, ..., z_r$, so the homogeneous polynomials of degree d, taken up to scalar factors, are parametrized by a projective space of dimension r that we denote by Z. Points of Z correspond bijectively to divisors of degree d in the projective plane (see Section **1.3.5**).

The product space $Z \times \mathbb{P}^2$ represents pairs (D, p), where D is a divisor of degree d and p is a point of $\mathbb{P}^2$. The variable homogeneous polynomial f may be written as $f(z, x)$. It is bihomogeneous, linear in z and of degree d in x. So the locus Γ: $\{f(z, x) = 0\}$ in $Z \times \mathbb{P}^2$ is a (Zariski) closed set whose points are pairs (D, p) such that p is a point of the divisor D. The set Σ of pairs (D, p) such that p is a singular point of D is also closed. It is defined by the system of equations $f_0(z, x) = f_1(z, x) = f_2(z, x) = 0$, where f_i is the partial derivative, as usual. The partial derivatives f_i are bihomogeneous, of degree 1 in z and degree $d - 1$ in x.

The next proposition isn't easy to prove directly, but the proof becomes easy when one uses the fact that projective space is proper.

singclosed **5.9.6. Proposition** *The singular divisors of degree d form a (Zariski) closed subset of the space Z of all curves of degree d.*

proof. Theorem 5.9.4 tells us that the image of the subset Σ via projection to Z is closed.. Its points correspond to singular divisors. $\square$

surfaceline **5.9.7. Example.** (*surfaces that contain a line*) We go back to the discussion of lines in a surface of Chapter 3. As in that discussion, let $\mathbb{S}$ denote the projective space that parametrizes surfaces of degree d in $\mathbb{P}^3$.

surfaceswith-line **5.9.8. Proposition** *In $\mathbb{P}^3$, the surfaces of degree d that contain a line form a closed subset of the space $\mathbb{S}$.*

proof. Let $\mathbb{G}$ be the Grassmanian $G(2,4)$ of lines in $\mathbb{P}^3$, and let Ξ be the subset of $\mathbb{G}\times\mathbb{S}$ of pairs of pairs $[\ell]$, $[S]$ such that $\ell\subset S$. Lemma 3.3.12 tells us that Ξ is a closed subset of $\mathbb{G}\times\mathbb{S}$. Therefore its image W in $\mathbb{S}$ is closed. $\square$

We now procede with the proof of Theorem 5.9.4. We will need to tweak the curve criterion for closed sets to prove it. We make use of Corollary 5.8.4 and the next lemma:

curvecrit-forpspace **5.9.9. Lemma.** *Let q be a point of a smooth affine cuve C, and let $C' = C - \{q\}$. Every morphism $C' \xrightarrow{f'} \mathbb{P}^n Z$ to a projective space extends uniquely to a morphism $C \xrightarrow{f} \mathbb{P}^n$.*

proof. Let K be the function field of C. The morphism f' gives us a point of $\mathbb{P}^n$ with values in K. Such points correspond bijectively, both to morphisms $C\to\mathbb{P}^n$ and to morphisms $C'\to\mathbb{P}^n$ (see Corollary 5.4.3). $\square$

proof of Theorem 5.9.4. We go back to the notation of Definition 5.9.3. We are given a diagram

$$\begin{array}{ccc} W & \longrightarrow & Z\times X \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\pi} \\ Y & \longrightarrow & Z \end{array}$$

in which X is a projective variety and W is a closed subset of $Z\times X$. The set Y is the image of W in Z, and the map σ is the restriction of π. We are to prove that Y is closed in Z. We may assume that X is a projective space. Also, we know that Y is constructible. It is a union of locally closed sets $Y_1,...,Y_k$. It suffices to show that, for $i=1,...,k$, the closure $\overline{Y}_i$ of Y_i is contained in Y. Let $\overline{W}_i=\pi^{-1}(\overline{Y}_i)$. This is a closed subset of $Z\times X$. If $\overline{W}_i$ maps surjectively to $\overline{Y}_i$ for each i, then W maps surjectively to $\overline{Y}=\overline{Y}_1\cup\cdots\cup\overline{Y}_k$, and therefore $\overline{Y}=Y$. So it suffices to prove the theorem when Y is open in its closure.

We apply the curve criterion. Suppose given a morphism $C\xrightarrow{f} X$ from a smooth affine curve to Z and a point q of C such that the image of $C'=C-\{q\}$ is contined in Y. We must show that $f(q)$ lies in Y. Corollary 5.8.4 tells us that there is a smooth affine curve D that fits into a diagram

$$\begin{array}{ccc} D & \xrightarrow{f'} & W \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle\tau} \\ C & \xrightarrow{f} & Z \end{array}$$

Let $\widetilde{D}$ be the normalization of C in the function field of D. This smooth affine curve comes with an integral, and therefore surjective, morphism $\widetilde{D}\xrightarrow{\widetilde{g}} C$.

We show that the morphism f' extends to a morphism $\widetilde{D}\xrightarrow{\widetilde{f}} W$: Let $\widetilde{D}\xrightarrow{d} Z$ denote the composed morphism $f\widetilde{g}$, and let $D\xrightarrow{h} X$ be the morphism obtained by restriction from the projection $Z\times X\to X$. Corollary 5.4.3 shows that h extends to a morphism $\widetilde{D}\xrightarrow{\widetilde{h}} X$. The pair of morphisms $(d,\widetilde{h})$ defines a morphism $\widetilde{D}\to Z\times X$ that extends the morphism $D\to Z\times X$. Since the image of D is in the closed set W, so is the image of $\widetilde{D}$. This gives us the morphism $\widetilde{D}\xrightarrow{\widetilde{f}} W$:

$$\begin{array}{ccc} \widetilde{D} & \xrightarrow{\widetilde{f}} & W \\ {\scriptstyle \widetilde{g}}\downarrow & & \downarrow{\scriptstyle \tau} \\ C & \xrightarrow{f} & Z \end{array}$$

Since the map $\widetilde{g}$ is surjective, the image of C is contained in the image Y of W. □

semicont 5.10 Fibre Dimension

A function $Y \xrightarrow{\delta} \mathbb{Z}$ from a variety to the integers is *constructible* if, for every integer n, the set of points of Y such that $\delta(p) = n$ is constructible, and δ is *upper semicontinuous* if for every n, the set of points such that $\delta(p) \geq n$ is closed. For brevity, we may refer to an upper semicontinuous function as *semicontinuous*, though the term is ambiguous. since a function might be lower semicontinuous.

If a function δ on a curve C is semicontinuous, it will be a constant c on a nonempty open subset U and its value on points not in U will be greater or equal to c.

The next curve criterion for semicontinuous functions follows from the criterion for closed subvarieties.

uppercrit **5.10.1. Proposition.** *(curve criterion for semicontinuity) A function $Y \xrightarrow{\delta} \mathbb{Z}$ is semicontinuous if and only if it is a constructible function, and for every morphism $C \xrightarrow{f} Y$ from an affine curve C to Y, the composition $\delta \circ f$ is a semicontinuous function on C.* □

Let Z be a closed subset of a variety X, and let p be a point of Z. The *local dimension of Z at p* is the maximum dimension among the irreducible components of Z that contain p. For example, in $\mathbb{P}^3$, let L be a line that meets a plane H at a point p, and let $Z = H \cup L$. The local dimension of Z at every point of H is 2, and is 1 at points of L different from p.

Let $Y \xrightarrow{f} X$ be a morphism, let q be a point of Y, and let F be the fibre of f over $p = f(q)$. The *fibre dimension* $\delta(q)$ of f at q is the local dimension of the fibre F at q.

uppersemi **5.10.2. Theorem.** *(semicontinuity of fibre dimension) Let $Y \xrightarrow{u} X$ be a morphism of varieties, and let $\delta(q)$ denote the fibre dimension at a point q of Y.*

(i) *Suppose that X is a smooth curve, that Y has dimension n, and that u is not a constant map from Y to a point of X. Then δ is the constant function $n-1$: Every fibre has constant dimension equal to $n-1$.*

(ii) *Suppose that the image of Y contains a nonempty open subset of X, and let the dimensions of X and Y be m and n, respectively. There is a nonempty open subset X' of X such that $\delta(q) = n - m$ for every point q in the inverse image of X'.*

(iii) *δ is a semicontinuous function on Y.*

The proof of this theorem makes a good exercise.