Chapter 4 INTEGRAL MORPHISMS

The concept of an algebraic integer was one of the most important ideas contributing to the development of algebraic number theory in the 19th century. Then in the 20th century, through the work of Noether and Zariski, its analog became essential in algebraic geometry. We study this analog here. Some of the things we discuss are:

If $A \subset B$ are domains, an element of B is *integral over* A if it is the root of a *monic* polynomial with coefficients in A, and B is an *integral extension* of A if every element of B is *integral over* A. If A and B are finite-type domains and B is integral over A, then B is a finite A-module. The Noether Normalization Theorem asserts that every finite-type domain is an integral extension of a polynomial ring.

Let K be the fraction field of a finite-type domain A. The *normalization* of A is the set of all elements of K that are integral over A. The normalization is finite A-module.

A morphism $Y \xrightarrow{u} X$ is a *finite morphism* if the inverse image Y' of every affine open set $X' = \operatorname{Spec} A$ of X is affine, and its coordinate ring is a finite A-module. Chevalley's Finiteness Theorem 4.5.2 asserts that if X and Y are projective and the fibres of u are finite sets, then u is a finite morphism.

We study double planes in the last section, and we relate a cubic surface to a double plane whose branch locus is a curve of degree 4. This allows us to determine the number of lines on a generic cubic surface in $\mathbb{P}^3$.

nakayama

Section 4.1 The Nakayama Lemma

(Tadasi Nakayama (1912-1964))

It won't be surprising that eigenvectors are important, but the way that they are used to study rings and modules may be new to you.

Let P be an $n \times n$ matrix with entries in a ring A. As usual, the characteristic polynomial of P is $p(t) = \det(tI - P)$. The concept of an eigenvector for P makes sense when the entries of a vector are in an A-module. A vector $v = (v_1, ..., v_n)^t$ with entries in a module is an *eigenvector* of P with *eigenvalue* λ if $Pv = \lambda v$. The assumption that an eigenvector must be nonzero isn't very useful when the entries are in a module, so we drop it.

eigenval **4.1.1. Lemma.** *Let p be the characteristic polynomial of an $n \times n$ matrix P. If v is an eigenvector of P with eigenvalue λ, then $p(\lambda)v = 0$.*

The usual proof, in which one multiplies the equation $(\lambda I - P)v = 0$ by the cofactor matrix of $\lambda I - P$, carries over. □

Here is the most important application of this lemma.

nakaya-malem **4.1.2. Nakayama Lemma.** Let M be a finite module over a ring A, and let J be an ideal of A. If $M = JM$, there is an element z in J such that $m = zm$ for all m in M, or such $(1-z)M = 0$.

Since the inclusion $M \supset JM$ is always true, the hypothesis $M = JM$ can be replaced by $M \subset JM$,

proof. Let $v = (v_1, ..., v_n)^t$ be a vector with entries in M and whose entries generate M. The equation $M = JM$ tells us that there are elements p_{ij} in J such that $v_i = \sum p_{ij} v_j$. In matrix notation, $v = Pv$. So v is an eigenvector of P with eigenvalue 1, and $p(1)v = 0$. Since the entries of P are in J, inspection of the matrix $I - P$ shows that $p(1)$ has the form $1 - z$, with z in J. Then $(1-z)v_i = 0$ for all i, and since $v_1, ..., v_n$ generate, $(1-z)M = 0$. □

idealzero **4.1.3. Corollary.** *Let A be a noetherian domain.*
(i) *If I and J are ideals of A and if $I = JI$, then either I is the zero ideal or J is the unit ideal.*
(ii) *Let J be an ideal of A that isn't the unit ideal. The intersection $\bigcap J^n$ of the powers of J is the zero ideal.*
(iii) *Let x and y be nonzero elements of a noetherian domain. The integers k such that x^k divides y are bounded.*

proof. **(i)** Suppose that $I = IJ$. Since A is noetherian, I is a finite A-module. The Nakayama Lemma tells us that there is an element z of J such that $zx = x$ for all x in I. If I isn't zero, we may choose a nonzero element x of I and cancel x from the equation $zx = x$, to obtain $z = 1$. Then 1 is in J, and J is the unit ideal.

(ii) The intersection $I = \bigcap J^n$ is an ideal, and it has the property that $I = JI$. Since J isn't the unit ideal, $I = 0$.

(iii) The intersection of the powers $x^k A$ of the ideal xA is the zero ideal. □

extnotunit **4.1.4. Corollary.** *Let $A \subset B$ be finite-type domains such that B is a finite A-module, and let J be an ideal of A. If the extended ideal JB is the unit ideal of B, then J is the unit ideal of A.*

proof. Suppose that $JB = B$. Since B is an integral extension of A, it is a finite A-module. The Nakayama Lemma applies. There is an element z in J such that $zb = b$ for all b in B. Since B is a domain, $z = 1$. So J is the unit ideal. □

fracnotfinite **4.1.5. Corollary.** *Let A be a subring of a field K. If K is a finite A-module, then A is a field.*

proof. Suppose that K is a finite A-module. Let x be nonzero element of A, and let J be the principal ideal xA. Since x is invertible in K, $JK = K$. Therefore J is the unit ideal, which shows that x is invertible in A. So every nonzero element of A is a unit, which means that A is a field. □

Since there are many subrings of fields that aren't fields themselves, we see that, in the Nakayama Lemma, the hypothesis that one is dealing with a finite module cannot be dropped.

int Section 4.2 Integral Extensions

An *extension* of a domain A is a domain B that contains A. An element β of an extension B is *integral over A* if it is a root of a monic polynomial with coefficients in A, say $f(\beta) = 0$, where

eqn (4.2.1)
$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0,$$

and a_i are in A. An extension of A is an *integral extension* if all of its elements are integral over A.

If $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ are affine varieties, and if $A \subset B$ is an integral extension, we call the morphism $Y \xrightarrow{u} X$ defined by the inclusion $A \subset B$ an *integral morphism.*

The next example is helpful for an intuitive understanding of the geometric meaning of integrality.

integralovercx **4.2.2. Example.** Let X denote the affine line $\operatorname{Spec} A$, $A = \mathbb{C}[x]$, and let Y be the plane affine curve defined by an irreducible polynomial $f(x,y)$. So $Y = \operatorname{Spec} B$, $B = \mathbb{C}[x,y]/(f)$. The inclusion of A into B gives us a morphism $Y \xrightarrow{u} X$, the restriction of the projection from the plane $\mathbb{A}^2_{x,y}$ to the line X.

We write f as a polynomial in y whose coefficients are polynomials in x:

$$f(x,y) = a_n(x)y^n + a_{n-1}(x)y^{n-1} + \cdots + a_0(x).$$

Let x_0 be a point of X, and let $\overline{a}_i = a_i(x_0)$, so that $f(x_0, y) = \overline{a}_n x^n + \overline{a}_{n-1}x^{n-1} + \cdots + \overline{a}_0$. The fibre of Y over x_0 is the set of points (x_0, y_0) such that y_0 is a root of $f(x_0, y)$. Because f is irreducible, the discriminant of f with respect to the variable y isn't identically zero (**??**). So for most x_0, $f(x_0, y)$ will have nonzero discriminant and therefore it will have n distinct roots.

If f is monic, the residue of y in B will be integral over A, and the polynomial $f(x, y)$ will have degree n for every x. The absolute values of the roots of a monic polynomial can be bounded in terms of the coefficients (see [Algebra], 12.4.10). So as x approaches a point x_0 at which the discriminant vanishes, some roots come together, but the roots remain bounded.

On the other hand, if the leading coefficient $a_n(x)$ isn't constant and if x_0 is a root of a_n, then $f(x_0, y)$ will have degree less than n. Above x_0, some roots are missing. What happens is that, as x approaches x_0, at least one root tends to infinity. (In calculus, one says that the locus $f(x, y) = 0$ has a vertical asymptote at x_0.) This is seen when one divides f by its leading coefficient. Let $c_i(x) = a_i(x)/a_n(x)$, and let

$$g(x, y) = y^n + c_{n-1}y^{n-1} + \cdots + c_n \quad \big(= f(x, y)/a_n \big)$$

The monic polynomial $g(x_0, y)$ has the same roots as $f(x_0, y)$ for all x_0 such that $a_n(x_0) \neq 0$. Suppose that $a_n(x_0) = 0$. Because f is irreducible, at least one coefficient coefficient of f, say a_i, doesn't have x_0 as root. Then the coefficient c_i tends to infinity as x approaches x_0. Since c_i is a symmetric function in the roots, the roots don't remain bounded.

This is the general picture: The roots of a polynomial vary continuously, and they remain bounded when the leading coefficient isn't zero. If the leading coefficient vanishes at a point, some roots are unbounded near that point. □

$$figure$$

The next lemma shows that one can always clear the denominator in an algebraic element to obtain one that is integral.

cleardenom **4.2.3. Lemma.** *Let A be a domain with fraction field K, let L be a field extension of K. and let β be an element of L that is algebraic over K. Say that β is a root of the polynomial $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, with $a_i \in K$. Then $\beta' = a_n\beta$ is integral over A.*

proof. β' is a root of $x^n + (a_n a_{n-1})x^{n-1} + (a_n^2 a_{n-2})x^{n-2} + \cdots + (a_n^n a_0)$. □

aboutintegral **4.2.4. Lemma.** *Let $A \subset B$ be domains.*

(i) *The ring extension $A[b]$ of A generated by an element b of B is a finite A-module if and only if b is integral over A.*

(ii) *The set of elements of B that are integral over A is a subring of B.*

(iii) *If B is generated by finitely many integral elements, it is a finite A-module.*

(iv) *Suppose that B is an integral extension of A. An element of an extension of B that is integral over B is also integral over A.* □

integralifffinite **4.2.5. Corollary.** *An extension $A \subset B$ of finite-type domains is an integral extension if and only B is a finite A-module.* □

The next theorem is named after Max Noether (1844-1921), the father of Emmy Noether. We will make use of it often.

noethernormal **4.2.6. Noether Normalization Theorem.** Let A be a finite-type algebra over an infinite field k. There exist elements $y_1, \ldots, y_n$ in A that are algebraically independent over k, and such that A is a finite module over the polynomial subalgebra $k[y_1, \ldots, y_n]$.

The theorem can be stated by saying that every affine variety X admits an integral morphism $X \to \mathbb{A}^n$ to an affine space.

nonzerocoeff **4.2.7. Lemma.** *Let k be an infinite field, and let $f(x)$ be a nonzero polynomial of degree d in $x_1, \ldots, x_n$, with coefficients in k. After a suitable linear change of variable, the coefficient of x_n^d in f will be nonzero.*

proof. We use a change of variable of the form

$$x_i = x'_i + c_i x'_n,$$

with elements c_i in k to be determined. Let h be the homogeneous part of f of degree d. The coefficient of ${x'}_n^d$ in the polynomial $f(x' + cx'_n)$ is obtained by substituting $x'_1 = \cdots = x'_{n-1} = 0$ and $x'_n = 1$ into $h(x' + cx'_n)$. It is $h(c_1, \dots, c_{n-1}, 1 + c_n)$, and it will be nonzero for most choices of c. □

proof of the Noether Normalization Theorem. Say that A is generated as algebra by the elements $x_1, \dots, x_n$. We use induction on n. If those elements are algebraically independent over k, A will be a polynomial ring, and we are done. If not, they will satisfy a polynomial relation $f(x) = 0$ of some degree d, with coefficients in k. The lemma tells us that, after a suitable change of variable, the coefficient of x_n^d in f will be nonzero. It can be normalized to 1. Then f will be a monic polynomial in x_n with coefficients in the subalgebra R generated by $x_1, \dots, x_{n-1}$. So x_n will be integral over R, and therefore A will be a finite R-module. By induction on n, we may assume that R is a finite module over a polynomial subalgebra P. Then A is a finite module over P too. □

nullfour **4.2.8. Nullstellensatz (version 4).** Let K be a field extension of an infinite field k, and suppose that K is a finite-type k-algebra. Then K is a finite extension of k (a finite-dimensional K-vector space).

proof. The Noether Normalization Theorem tells us that K is a finite module over a polynomial subalgebra $P = k[y_1, \dots, y_d]$, and Corollary 4.1.5 shows that P is a field. This implies that $d = 0$. So K is a finite module over k. □

truefinfld **Note.** Theorems 4.2.6 and 4.2.8 are true when k is a finite field.

finint Section 4.3 Finiteness of the Integral Closure

Let A be a domain with fraction field K, and let L be a finite field extension of K.

The *integral closure of A in L* is the set of all elements of L that are integral over A. The integral closure is a ring (Lemma 4.2.4 **(ii)**).

The *normalization* $\overline{A}$ of A is the integral closure of A in K – the set of all elements of the fraction field K that are integral over A. A *normal domain* A is a domain that is equal to its normalization. A *normal variety* X is a variety that has an affine covering $\{X_i = \operatorname{Spec} A_i\}$ in which A_i are normal domains.

If $\overline{A}$ is the normalization of a finite-type domain A, and if $X = \operatorname{Spec} A$ and $\overline{X} = \operatorname{Spec} \overline{A}$, we call $\overline{X}$ the *normalization* of X.

The object of this section is to prove the next theorem:

normalfinite **4.3.1. Theorem.** *Let A be a finite-type domain with fraction field K of characteristic zero, and let L be a finite field extension of K. The integral closure of A in L is a finite A-module, and therefore a finite-type domain. In particular, the normalization of A is a finite A-module and a finite-type domain.*

The theorem is also true for a finite-type k-algebra when k is a field of characteristic p, though the proof we give here doesn't work.

nodecurve **4.3.2. Example.** *(normalization of a nodal cubic curve)* The algebra $R = \mathbb{C}[u, v]/(v^2 - u^3 - u^2)$ embeds into the one-variable polynomial algebra $S = \mathbb{C}[x]$ by $u = x^2 - 1$ and $v = x^3 - x$. Then $x = v/u$, so the fraction fields of the two algebras are equal, and the equation $x^2 - u - 1 = 0$ shows that x is integral over R. Here S is the normalization of R.

The curve $C = \operatorname{Spec} R$ has a node at the origin, $\operatorname{Spec} S$ is the affine line $\mathbb{A}^1_x$, and the inclusion $S \subset R$ defines an integral morphism $\mathbb{A}^1_x \to C$. The fibre of this morphism over the point $(0, 0)$ of C is the point pair $x = \pm 1$, and the morphism is bijective at all other points. One may regard C as the variety obtained from the affine line by gluing the points $x = \pm 1$ together. □

##figure##

ufdnormal **4.3.3. Lemma.** **(i)** *A unique factorization domain is normal. In particular, a polynomial ring over a field is normal.*

(ii) *Let R be a normal domain, and let s be a nonzero element of R. The localization R_s is normal.*

(iii) *If $s_1, ..., s_k$ are elements of a domain R that generate the unit ideal, and if the localizations R_{s_i} are normal for every i, then R is normal.*

proof. **(i)**: Let R be a unique factorization domain, and let α be an element of its fraction field K that is integral over R. Say that

$$\alpha^n + a_1\alpha^{n-1} + \cdots + a_{n-1}\alpha + a_n = 0,$$

with a_i in R. We write $\alpha = r/s$, where r and s are relatively prime elements of R. Multiplying by s^n gives us the relation $r^n + a_1r^{n-1}s + \cdots + a_ns^n = 0$, or

$$r^n = -s\,(a_1r^{n-1} + \cdots + a_ns^{n-1}).$$

This equation shows that if a prime element p of R divides s, it also divides r. Since r and s are relatively prime, there is no such element. Therefore s is a unit, and α is in A.

We omit the verification of **(ii)** and **(iii)**. □

abouttracetwo **4.3.4. Lemma.** *Let A be a normal noetherian domain with fraction field K of characteristic zero, and let β be an element of a field extension L of K that is integral over A. The coefficients of the (monic) irreducible polynomial f for β over K are elements of A.*

proof. Since we may replace L by $K(\beta)$, we may assume that L is a finite extension of K. A finite extension embeds into a Galois extension, so we may assume that L is a Galois extension of K. Let G be its Galois group, and let $\{\beta_1, \ldots, \beta_r\}$ be the G-orbit of β, with $\beta = \beta_1$. The irreducible polynomial for β over K is

orbitpoly **(4.3.5)**

$$f(x) = (x - \beta_1) \cdots (x - \beta_r).$$

Its roots are the elements of the orbit, and its coefficients are symmetric functions in the roots. If β is integral over A, then all elements of the orbit are integral over A, and therefore the symmetric functions are integral over A. The symmetric functions are in K, and since A is normal, they are elements of A. So the coefficients of f are in A. □

Let L/K be a finite field extension, and let β be an element of L. When L is viewed as a vector space over K, multiplication by β becomes a linear operator on L. The *trace* of this operator will be denoted by $\mathrm{tr}(\beta)$. The trace is a K-linear map, a linear transformation, $L \to K$.

abouttraceone **4.3.6. Lemma.** *Let β be an element of a finite field extension L of K, and let*

$$f(x) = x^r + a_1x^{r-1} + \cdots + a_r$$

be the irreducible polynomial for β over K. Let $n = [L : K]$ and $d = [L : K(\beta)]$, so that $n = dr$. Then $\mathrm{tr}(\beta) = -da_1$. If β is an element of K, then $\mathrm{tr}(\beta) = n\beta$.

proof. With respect to the basis $1, \beta, \ldots, \beta^{r-1}$, the matrix of multiplication by β on $K(\beta)$ will have the form illustrated below for $n = 3$. Its trace is $-a_1$.

$$M_\beta = \begin{pmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{pmatrix}.$$

Next, let $(u_1, \ldots, u_d)$ be a basis for L over the intermediate field $K(\beta)$. Then $\{\beta^iu_j\}$, with $i = 0, \ldots, r-1$ and $j = 1, \ldots, d$, will be a basis for L over K. When this basis is listed in the order

$$(u_1, u_1\beta, ..., u_1\beta^{n-1};\ ,\ldots\ ; u_d, ..., u_d\beta^{n-1}),$$

the matrix of multiplication by β will be made up of d blocks of the matrix M_β. □

formnondeg **4.3.7. Lemma.** *Let A be a normal noetherian domain with fraction field K* of characteristic zero*, and let L be a finite field extension of K. The form $L \times L \to K$ defined by $\langle \alpha, \beta \rangle = \mathrm{tr}(\alpha\beta)$ is K-bilinear, symmetric, and nondegenerate. If α and β are integral over A, then $\langle \alpha, \beta \rangle$ is an element of A.*

proof. The form is obviously symmetric, and it is bilinear because trace is linear. A form is nondegenerate if its nullspace is zero, which means that when α is nonzero, there is an element β such that $\langle \alpha, \beta \rangle \neq 0$. We let $\beta = \alpha^{-1}$. Then $\langle \alpha, \beta \rangle = \mathrm{tr}(1)$, which is the degree $[L : K]$ of the field extension. It is here that the hypothesis on the characteristic of K enters: The degree is a nonzero element of K. Finally, if α and β are integral over A, so is their product $\alpha\beta$ (4.2.4)**(ii)**. Lemmas 4.3.4 and 4.3.6 show that $\langle \alpha, \beta \rangle$ is an element of A. □

proof of Theorem 4.3.1. Let A be a finite-type domain with fraction field K, and let L be a finite field extension of K. We are to show that the integral closure of A in L is a finite A-module.

Step 1: We may assume that A is normal.

We use the Noether Normalization Theorem to write A as a finite module over a polynomial subring $R = \mathbb{C}[y_1, \ldots, y_d]$. Let F be the fraction field of R. Then K and L are finite extensions of F. An element of L will be integral over A if and only if it is integral over R (4.2.4)**(iv)**. So the integral closure of A in L is the same as the integral closure of R in L. We may therefore replace A by the normal algebra R, and K by the field F.

Step 2: Bounding the integral extension.

Let $(v_1, \ldots, v_n)$ be a K-basis for L whose elements are integral over the normal domain A (see Lemma 4.2.3). We define a K-linear map

mapvector (4.3.8) $$T : L \to K^n$$

by $T(\beta) = \big(\langle \beta, v_1 \rangle, \ldots, \langle \beta, v_n \rangle\big)$, where $\langle\ ,\ \rangle$ is the form defined in Lemma 4.3.7. If $\langle \beta, v_i \rangle = 0$ for all i, then because $(v_1, \ldots, v_n)$ is a basis, $\langle \beta, \gamma \rangle = 0$ for all γ in L, and since the form is nondegenerate, $\beta = 0$. Therefore T is injective.

Let B be the integral closure of A in L. The basis elements v_i are in B, and if β is an element of B, the elements βv_i will be in B too. Therefore $\langle \beta, v_i \rangle$ will be in A, and $T(\beta)$ will be in A^n. When we restrict T to B, we obtain an injective map $B \to A^n$ that we denote by T_0. Since T is K-linear, T_0 is a A-linear. It maps B isomorphically to its image, an A-submodule of A^n. Since A is noetherian, every submodule of the finite A-module A^n is finitely generated. So the image is a finite A-module, and B is a finite A-module too. □

prmint Section 4.4 Geometry of Integral Morphisms

The main facts about integral morphisms of affine varieties are summarized in Theorem 4.4.2, which shows that their geometry is as nice as could be expected for maps that, most often, aren't injective.

Let

finmorph (4.4.1) $$Y \xrightarrow{u} X$$

be an integral morphism of affine varieties, say $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$. We say that a closed subvariety D of Y *lies over* a closed subvariety C of X if C is the image of D, and we say that a prime ideal Q of B *lies over* a prime ideal P of A if P is the contraction $Q \cap A$ of Q. If C and D are the subvarieties defined by ideals P and Q of A and B, respectively, then D lies over C if and only if Q lies over P. For example, if a point y of $Y = \operatorname{Spec} B$ has image x in X, the maximal ideal $\mathfrak{m}_y$ lies over the maximal ideal $\mathfrak{m}_x$.

closedimage **4.4.2. Theorem.** *Let $Y \xrightarrow{u} X$ be an integral morphism of affine varieties.*

(i) *u is surjective, and its fibres have bounded cardinality.*

(ii) *The image of a closed subset of Y is closed in X.*

(iii) *The set of closed subvarieties of Y that lie over a closed subvariety C of X is finite and nonempty.*

incompar **4.4.3. Lemma.** **(i)** *Let $Y \xrightarrow{u} X$ be an integral morphism of affine varieties. If $D' \subset D$ are closed subvarieties of Y that lie over the same closed subvariety C of X, then $D' = D$.*

(ii) *Let $A \subset B$ be an integral extension of finite-type domains. If Q is a nonzero prime ideal of B, its contraction $P = Q \cap A$ is a nonzero prime ideal of A.*

proof. **(ii)** A nonzero element β of Q will be integral over A, say $\beta^n + a_{n-1}\beta^{n-1} + \cdots + a_0 = 0$, with $a_i \in A$. If $a_0 = 0$, then because B is a domain, we can cancel β from the equation. So we may assume $a_0 \neq 0$. The equation shows that a_0 is in Q, and since it is also in A, it is in P.

(i) The coordinate rings of C and D are $\overline{A} = A/P$ and $\overline{B} = B/Q$, where P and Q are prime ideals of A and B, respectively. Moreover, $\overline{B}$ is an integral extension of $\overline{A}$. So we may replace X and Y by the affine varieties C and D. Then what we must show is that if D' is a proper closed subset of Y, its image C' is a proper closed subset of X, or, if Q' is a nonzero prime ideal of B, then, then its contraction $P' = Q' \cap A$ is nonzero. This is **(ii)**. □

proof of Theorem 4.4.2 **(i)**. *(bounding the fibres)*

Let $\mathfrak{m}_x$ be the maximal ideal at point x of X. Corollary 4.1.4 shows that the extended ideal $\mathfrak{m}_x B$ is not the unit ideal of B, so it is contained in a maximal ideal of B, say $\mathfrak{m}_y$, where y is a point of Y. Then x is the image of y (**??**), so u is surjective.

Let $k(x)$ be the residue field of A at x. Then $\overline{B} = B/\mathfrak{m}_x B$ is a $k(x)$-algebra. Its maximal ideals correspond to the maximal ideals of B that contain $\mathfrak{m}_x B$, the ones that correspond to points y such that $u(y) = x$. Since B is a finite A-algebra, $\overline{B}$ is a finite-dimensional complex vector space. Proposition **??** tells us that $\overline{B}$ has finitely many maximal ideals. So there are finitely many points of Y that lie over x. □

proof of Theorem 4.4.2 **(ii)**. *(the image of a closed set is closed)*

It suffices to show that the image of a closed subvariety is closed. Let Q be the prime ideal of B that corresponds to a closed subvariety D of Y, and let C be the variety that corresponds to its contraction $P = Q \cap A$. The morphism $D \to C$ is integral, and by **(i)**, the map $D \to C$ is surjective. So C is the image of D.□

proof of Theorem 4.4.2. **(iii)** *(subvarieties that lie over a closed subvariety)*

The inverse image $Z = u^{-1}C$ of a closed subvariety C is closed in Y. It is the union of finitely many irreducible closed subsets, say $Z = \bigcup D_i'$. Let C_i' be the image of D_i'. Part **(ii)** tells us that C_i' closed in X. Since u is surjective, $C = \bigcup C_i'$, and since C is irreducible, it is equal to at least one C_i'. The components D_i' of Z such that $C_i' = C$ are the ones that lie over C.

Next, any subvariety D that lies over C will be contained in $Z = \bigcup D_i'$, and since it is irreducible, it will be contained in D_i' for some i. Lemma 4.4.3 shows that $D = D_i'$. □

Section 4.5 Chevalley's Finiteness Theorem

finmorph

A morphism of varieties $Y \xrightarrow{u} X$ is a *finite morphism* if the inverse image Y' of every affine open subset $X' = \operatorname{Spec} A$ of X is affine, say $Y' = \operatorname{Spec} B$, and B is a finite A-module. Integral morphisms of affine varieties and inclusions of a closed subvarieties into a variety X are examples of finite morphisms.

onecoverfinite

4.5.1. Lemma. *Let $\{X^i\}$ be a open covering of a variety X. If $Y \xrightarrow{u} X$ is a morphism of varieties whose restriction to each X^i is a finite morphism, then u is a finite morphism.*

proof. When we restrict a finite morphism $Y \to X$ to an open subvariety X' of X, the resulting map $Y' \to X'$ will be a finite morphism. Let $Y \xrightarrow{u} X$ be given, and suppose that there is a covering of X by open sets X^i to which the restrictions of u are finite morphisms. Any open subset of X can be covered by open subsets, each of which is a subset of the sets X^i, so any open subset of X can be covered by open subsets to which the restriction of u is a finite morphism.

Let L denote the function field of Y. We are to show that the restriction of u to any affine open subset of X is a finite morphism. So we may assume that X is affine, say $X = \operatorname{Spec} A$. Let $Y' = \operatorname{Spec} B'$ be a (nonempty) affine open subset of Y. The morphism u restricts to a morphism $Y' \to X$, which is defined by a ring homomorphism $A \xrightarrow{\varphi} B'$. The kernel of φ is independent of Y', because it is also the kernel of the composed map $A \to B' \subset L$. If $\overline{A}$ is the image of A in L, the morphism u will send every affine open set Y' to $\overline{X} = \operatorname{Spec} \overline{A}$. So u has image in $\overline{X}$. We may therefore replace X by $\overline{X}$, which reduces us to the case that φ is injective.

Since the simple localizations of X form a basis for the topology, we may cover X by simple localizations to which the restriction of u is finite. Thus there will be nonzero elements $s_1, ..., s_k$ that generate the unit ideal of A, such that, if $A_i = A_{s_i}$, $X^i = \operatorname{Spec} A_i$, and $u^{-1}(X^i) = Y^i$, then Y^i is affine, say $Y^i = \operatorname{Spec} B_i$, and B_i is a finite A_i-module. Let $B = \bigcap B_j$.

The plan is to show that $Y = \operatorname{Spec} B$. Let $A_{ij} = A_{s_is_j}$, and $X^{ij} = \operatorname{Spec} A_{ij}$. Then $Y^{ij} = u^{-1}(X^{ij})$ is a localization of Y^i and of Y^j, the spectrum of the ring $B_{ij} = B_i[s_j^{-1}] = B_j[s_i^{-1}]$. The localization $B[s_i^{-1}]$ of B is equal to the intersection of the localizations $B_j[s_i^{-1}]$, all of which contains B_i, and one of which, namely $B_i[s_i^{-1}]$ is equal to B_i. So the intersection of the localizations $B_j[s_i^{-1}]$ is B_i.

We choose a finite set $b_1, ..., b_n$ of elements of B that generates B_i as A_i-module for every i. We can do this because B_i is a finite A_i-module. Let C be the cokernel of the map $A^n \to B$ that sends $e_\nu \leadsto b_\nu$. The localization C_i of C is the cokernel of the map $A_i^n \to B_i$ (**??**), and is zero for every i. Therefore $C = 0$. So B is a finite A-module. According to Proposition **??**, there is a morphism $Y \xrightarrow{u} \operatorname{Spec} B$. Because the localization of u is an isomorphism for every i, u is an isomorphism. □

chevfin **4.5.2. Chevalley's Finiteness Theorem.** *Let Y be a closed subvariety of a product $\mathbb{P}^n \times X$ of a projective space with a variety X, and let π be the projection from Y to X. If the fibres of π are finite sets, then π is a finite morphism.*

projchevfin **4.5.3. Corollary.** *A morphism $Y \xrightarrow{u} X$ of projective varieties whose fibres are finite sets is a finite morphism.*

This corollary follows from the theorem when one replaces Y by the graph of the morphism u. □

proof of the Chevelley Finiteness Theorem.

This is Schelter's proof. Descending induction on Y (**??**) allows us to assume that for every proper closed subvariety V of Y, the restriction of π to V is a finite morphism. Lemma 4.5.1 shows that we may assume X affine, say $X = \operatorname{Spec} A$.

Let $y_0, ..., y_n$ be coordinates in $\mathbb{P}^n$, and let U^i be the standard affine open set $\{y_0 \neq 0\}$. To simplify notation, we replace the symbol $\times X$ by a tilde, writing $\widetilde{\mathbb{P}}$ for $\mathbb{P}^n \times X$, and $\widetilde{U}$ for $U \times X$, etc.

We first consider a special case, that there exists a hyperplane H in $\mathbb{P}^n$ such that Y is disjoint from $\widetilde{H} = H \times X$. We adjust coordinates so that H is the hyperplane H^0, and we let $Z = \widetilde{H}^0$. Then $Y \cap Z = \emptyset$ and Y is contained in $\widetilde{U}^0$.

Let $u_i = y_i/y_0$ and $v_i = y_i/y_1$ be coordinates in U^0 and U^1, respectively, with

$$u_0 = 1\,, \quad v_1 = 1\,, \quad \text{and} \quad v_0u_1 = 1$$

So $\widetilde{U}^0 = \operatorname{Spec} A[u_0, ..., u_n]$ and $\widetilde{U}^1 = \operatorname{Spec} A[v_0, ..., v_n]$. Since Y is closed in $\widetilde{\mathbb{P}}$, it is closed in $\widetilde{U}^0$. Therefore it is an affine variety, the zero set of a prime ideal P_0 of $A[u]$, and its coordinate algebra will be $B = A[u]/P_0$.

Next, we look on the standard affine open set U^1. Let Y^1 and Z^1 denote the closed subvarieties $Y \cap \widetilde{U}^1$ and $Z \cap \widetilde{U}^1$ of $\widetilde{U}^1$, respectively. Then Y^1 is the zero set of a prime ideal P_1 of $A[v]$, and Z^1 is the zero set of the principal ideal of $A[v]$ generated by v_0. The intersection $Y^1 \cap Z^1$ is empty because $Y \cap Z$ is empty, so the sum $P_1 + (v_0)$ is the unit ideal. There is an equation in $A[v]$ of the form

fplusg (4.5.4)
$$f_1(v) + g_1(v)v_0 = 1$$

with $f_1(v)$ in P_1 and $g_1(v)$ in $A[v]$.

This equation is also valid in the coordinate algebra of the intersection $\widetilde{U}^0 \cap \widetilde{U}^1$, which is the spectrum of the common localization $A[u, v] = A[v][v_0^{-1}] = A[u][u_1^{-1}]$. In $A[u, v]$, we may write the equation (4.5.4) in terms of u, using the relation

$$v_j = u_j u_1^{-1}$$

When we do this, and multiply by a large power u_1^k to clear denominators, we will obtain an equation in $A[u]$ of the form

$$F_1(u) + G_1(u) = u_1^k$$

where $F_1(u) = f_1(v)u_1^k$ and $G_1(u) = \big(g_1(v)v_0\big)u_1^k$. The ideals P_0 of $A[u]$ and P_1 of $A[v]$ generate the same ideal in $A[u, v]$. Since $f_1(v)$ is in P_1, $F_1(u)$ will be in P_0 if k is large enough.

Now the important point is this: As functions of u, the variables v_j have degree zero. Therefore $F_1(u)$ will be a polynomial of degree k, but because $v_0u_1 = 1$, $G_1(u)$ will have degree $k - 1$. When we restrict to Y, the term F_1 drops out, and we obtain the equation in B:

$$u_1^k = G_1(u)$$

in which G_1 has degree $k-1$.

We can replace U^1 by U^i for every index $i=1,...,n$, using the same large exponent k. Thus there will be relations in B of the form

uhatk (4.5.5)
$$u_i^k = G_i(u)$$

with G_i of degree $k-1$.

Suppose that an element β of B is represented by a polynomial $p(u)$ in $A[u]$. If a monomial m that appears in p is divisible by u_i^k for some i, say $m = u_i^k z$, then β is also represented by the polynomial obtained by substituting $G_i(u)z$ for m into $p(u)$, and $G_i(u)z$ has lower degree than m. By making such substitutions finitely often, we will be left with a polynomial $\widetilde{p}(u)$ that still represents β, and in which no monomial that appears is divisible by any u_i^k. Any monomial of degree $\geq nk+1$ will be divisible by u_i^k or at least one i, so the polynomial $\widetilde{p}$ will have degree at most nk. Therefore the monomials in u of degree $\leq nk$ span B as A-module. So B is a finite A-module.

This takes care of the case in which there exists a hyperplane H such that Y is disjoint from $\widetilde{H}$. The next lemma shows that we can cover the given variety X by open subsets to which this special case applies. Then Lemma 4.5.1 completes the proof.

avoidhplane **4.5.6. Lemma.** *Let hypotheses be as in the statement of Chevalley's Theorem. For every point p of X, there is a hyperplane H in $\mathbb{P}^n$ and an affine open neighborhood X' of p whose inverse image Y' in Y is disjoint from $\widetilde{H}$.*

proof. The fibre of Y over a point p of X will be finite a finite set of points $\widetilde{q}_1,...,\widetilde{q}_r$. Since $Y\subset\mathbb{P}^n\times X$ we can project these points to $\mathbb{P}^n$, obtaining a finite set $q_1,...,q_r$. We choose a hyperplane H in $\mathbb{P}^n$ that avoids this finite set. Then $\widetilde{H}$ avoids the fibre of Y over p. Let V denote the closed subset $Y\cap\widetilde{H}$ of Y. Since V is a proper closed subset of Y, every component of V is finite over X, and therefore has a closed image (Theorem 4.4.2). This is our induction hypothesis. Thus the image W of V in X is closed, and it doesn't contain p. Then $X'=X-W$ is the required neighborhood of p: If q' is a point of its inverse image Y', then $q'\notin V$, and therefore $q'\notin\widetilde{H}$. So $Y'\cap\widetilde{H}=\emptyset$. □

dplane Section 4.6 Example: Double Planes

affdplanes (4.6.1) affine double planes

Let A denote the polynomial algebra $\mathbb{C}[x,y]$, and let X denote the affine plane $\operatorname{Spec} A$. An *affine double plane* is a locus of the form $w^2=f(x,y)$ in affine 3-space, where f is a square-free polynomial – a nonconstant polynomial with no square factor. Let $B=\mathbb{C}[w,x,y]/(w^2-f)$, so that the double plane is $Y=\operatorname{Spec} B$. Let's denote by w,x,y both the variables and their residues in B.

dplanering-normal **4.6.2. Lemma** *The algebra B is a normal domain, and a free A-module with basis $(1,w)$. It has an automorphism σ of order 2, defined by $a+bw\rightsquigarrow a-bw$, and the algebra of σ-invariant elements is A.* □

The inclusion $A\subset B$ gives us an integral morphism $Y\xrightarrow{u}X$ that sends (w,x,y) to (x,y). Given a point $p=(x_0,y_0)$ of X, the equation $w^2=f(x_0,y_0)$ determines the points of Y over p. When $f(x_0,y_0)=0$, the only solution is $w_0=0$, and when $f(x_0,y_0)\neq 0$, there are two solutions $\pm w_0$. The reason that Y is called a *double plane* is that most points of the plane X are covered by two points of Y. Points of X correspond bijectively to σ-orbits of points of Y.

The curve $\{f=0\}$ in X, which we denote by Δ, is the *branch locus* of the covering.

We study prime ideals of B that lie over principal prime ideals of A. A prime ideal Q that lies over a principal prime ideal P needn't be a principal ideal, but its zero set D in Y will lie over the plane curve C defined by P, and D will be called a curve too.

We leave the proof of the next lemma as an exercise.

primesind-plane **4.6.3. Lemma.** *Let g be an irreducible element of the polynomial algebra A, let P be the principal prime ideal gA of A, and let C be the curve of zeros of g in the affine plane X. There are three possibilities:*

Case 1: g doesn't divide f. Then either

P **splits:** *There are two closed subvarieties of Y that lie over C, and there are two prime ideals of B that lie over P, or*

P **remains prime:** *There is one closed subvariety of Y that lies over C. The extended ideal $PB = gB$ is the unique prime ideal that lies over P.*

Case 2: g divides f. Then

P **ramifies:** *There is one closed subvariety of Y that lies over C. The extended ideal gB is not a prime ideal, but its radical $Q = (g, w)B$ is the unique prime ideal that lies over P.*

circleex-ample **4.6.4. Example.** Let $f(x, y) = x^2 + y^2 - 1$. The double plane $Y = \{w^2 = x^2+y^2-1\}$ is an affine quadric. Its branch locus Δ is the curve $\{x^2+y^2 = 1\}$.

The line $C_1 : \{y = 0\}$ in X meets the branch locus Δ transversally at the points $(\pm 1, 0)$, and the prime ideal yA remains prime, because $B/yB \approx \mathbb{C}[w, x]/(w^2-x^2+1)$ and w^2-x^2+1 is an irreducible polynomial. On the other hand, the line $C_2 : \{y = 1\}$ is tangent to Δ at the point $(0, 1)$, and it splits. When we set $y = 1$ in the equation for Y, we obtain $w^2 = x^2$. The locus $\{w^2 = x^2\}$ is the union of the two lines $\{w = x\}$ and $\{w = -x\}$ that lie over C_1.

##figure##

This example illustrates a general principle: A curve that intersects the branch locus transversally at some point doesn't split. We explain this now.

localanal **(4.6.5) local analysis**

Let's suppose that a plane curve $C : \{g = 0\}$ and the branch locus $\Delta : \{f = 0\}$ of a double plane meet at a point p. We adjust coordinates so that p becomes the origin $(0, 0)$, and we write

$$f(x, y) = \sum a_{ij}x^i y^j = a_{10}x + a_{01}y + a_{20}x^2 + \cdots$$

Since Δ contains the origin, the constant coefficient of f is zero. The line $\{a_{10}x+a_{01}y = 0\}$ is the tangent line to Δ at p. It is defined if the two linear coefficients aren't both zero. Similarly, writing $g(x, y) = \sum b_{ij}x^i y^j$, the tangent line to C, if defined, is the line $\{b_{10}x + b_{01}y = 0\}$.

Let's suppose that the two tangent lines are defined and distinct – that the curves intersect transversally at p. We change coordinates once more, to make the two tangent lines the coordinate axes. After adjusting by scalar factors, the polynomials f and g will have the form

$$f(x, y) = x + u(x, y) \quad \text{and} \quad g(x, y) = y + v(x, y),$$

where u and v are polynomials all of whose terms have degree at least 2.

Working in the classical topology in a neighborhood of p, the substitution $x_1 = x + u$ and $y_1 = y + v$ is invertible analytically, because the Jacobian matrix

jacob (4.6.6)
$$\left(\frac{\partial(x_1, y_1)}{\partial(x, y)}\right)_{(0,0)}$$

at p is invertible. It is the identity matrix. When we make this substitution, Δ becomes the locus $\{x_1 = 0\}$ and C becomes the locus $\{y_1 = 0\}$. In this local coordinate system, the equation $w^2 = f$ that defines the double plane becomes $w^2 = x_1$. When we restrict it to C by setting $y_1 = 0$, x_1 becomes a local coordinate function on C, and the restriction of the equation remains $w^2 = x_1$. The inverse image Z of C doesn't decompose, locally at p. Therefore it doesn't decompose globally either, and this shows that P remains prime.

splitnot-transversal **4.6.7. Corollary.** *A curve C that splits cannot meet the branch locus transversally at any point.* □

This isn't a complete analysis. When C and Δ are tangent at every point of intersection, C may split or not, and which possibility occurs cannot be decided by a local analysis in most cases. However, there is one case in which a local analysis suffices to decide splitting, the case that C is a line. Say that $C \approx \operatorname{Spec}\mathbb{C}[t]$. We restrict the polynomial f to C, obtaining a polynomial $\phi(t)$ in t. A root of ϕ corresponds to an intersection of the line C with Δ, and a multiple root corresponds to an intersection at which C and Δ are tangent, or at which Δ is singular. The line C will split if and only if $\phi(t)$ is a square in $\mathbb{C}[t]$, and this will be true if and only if every root of ϕ has even multiplicity. If the multiplicity r_i of every root a_i is even, then $\phi(t) = c(t-a_1)^{r_1}\cdots(t-a_k)^{r_k}$ will be a square.

tangentsplit **4.6.8. Corollary.** *A line in the plane X splits if it has a simple tangency with the branch locus at every intersection point.* □

A *rational curve* is a curve whose function field is a rational function field $\mathbb{C}(t)$ in one veriable. One may make a similar analysis for any rational plane curve, such as a conic, but one needs to examine its singular points and its points at infinity as well as its points at finite distance.

curiouspoint **(4.6.9) a curious point.**

Most curves C in the plane X will intersect a given branch locus $\Delta : \{f = 0\}$ transversally, and therefore won't split. In fact, at first glance it isn't obvious that there will be *any* curves in X that split, when f has high degree. However, every curve in Y lies over a curve in X, and most curves in Y won't be symmetric with respect to the symmetry σ that sends $(w, x, y) \rightsquigarrow (-w, x, y)$. For example, when we slice Y by a plane passing through a point $q = (w_0, x_0, y_0)$, the slice will most often not contain the point $q' = (-w_0, x_0, y_0)$. Then if D is the component of the slice that contains q, $D\sigma$ will contain q', and it will be distinct from D. But the images in X of D and $D\sigma$ will be the same. The image will split. Curves that split do exist.

Here is the curious point: Let D be a curve in Y that lies over a curve C in X. Then C won't split if it has a transversal intersection with Δ, and this will be true for most curves in X. On the other hand, when C regarded as the image of the curve D on Y, C will split unless D is symmetric with respect to σ, and most curves in Y won't be symmetric.

			"most" curves on Y
	curves on Y	(D *symmetric*)	(D *not symmetric*)
curious **(4.6.10)**	↓	↓	↓
	curves on X	(C *doesn't split*)	(C *splits*)
		"most" curves on X	

One has to be careful about the meaning of the word "most".

projdplane **(4.6.11) projective double planes**

A projective double plane is a locus of the form

wtdplane (4.6.12)
$$y^2 = f(x_0, x_1, x_2),$$

where f is a square-free homogeneous polynomial of even degree $2d$. To regard this as a homogeneous equation, so we must assign *weight* d to the variable y. Then, since we have weighted variables, we must work in a *weighted projective space* $\mathbb{WP}$ with coordinates x_0, x_1, x_2, y, where x_i have weight 1 and y has weight d. A point of this weighted space $\mathbb{WP}$ is represented by a nonzero vector (x_0, x_1, x_2, y) with the relation that, for all $\lambda \neq 0$, $(x_0, x_1, x_2, y) \sim (\lambda x_0, \lambda x_1, \lambda x_2, \lambda^d y)$. The points of the weighted projective space $\mathbb{WP}$ that solve the equation (4.6.12) are the points of the *projective double plane* Y.

If (x, y) solves (4.6.12) and $(x) = (0, 0, 0)$, then $y = 0$ too. The vector $(0, 0, 0, 0)$ doesn't represent a point of $\mathbb{WP}$. Therefore the projection $\mathbb{WP} \to \mathbb{P}^2 = X$ that sends $(x, y) \rightsquigarrow x$ is defined at all points of Y, and maps Y to the projective plane X. The fibre of Y over the point x of X consists of the points (x, y) and $(x, -y)$, which will be equal if and only if x lies on the *branch locus* of the double plane, the (possibly reducible) curve $\Delta : \{f = 0\}$ in X. The map $\sigma : (x, y) \rightsquigarrow (x, -y)$ is an automorphism of Y, and points of X correspond bijectively to σ-orbits in Y.

Since the double plane Y is embedded into a weighted projective space, it isn't presented to us as a projective variety in the usual sense. However, it can be embedded into a projective space in the following way: The projective plane $\mathbb{P}^2_x$ can be embedded by a Veronese embedding of higher order, using as coordinates the monomials $m_1, m_2, \dots$ of degree d in x_0, x_1, x_2. This embeds $\mathbb{P}^2$ into a projective space $\mathbb{P}^N$ where $N = \binom{d+2}{2} - 1$. When we add one more coordinate y to this embedding, we obtain an embedding of the weighted projective space $\mathbb{WP}$ into $\mathbb{P}^{N+1}$, that sends the point (x, y) to (m, y). The double plane Y can be realized as a projective variety by this embedding.

If $Y \to X$ is a projective double plane then, as happens with affine double planes, a curve C in X may split in Y or not. If C has a transversal intersection with the branch locus Δ, it will not split, while if C is a line that has an ordinary tangent to the branch locus Δ at every intersection point, it will split (Corollary 4.6.8). For example, if the branch locus Δ is a generic quartic curve, the lines that split will be the bitangent lines (see Section **??**).

homogdplan **(4.6.13) homogenizing an affine double plane**

We construct a projective double plane by homogenizing an affine double plane. Let's write an affine double plane as

relabelaffinedplane (4.6.14)
$$w^2 = F(u_1, u_2).$$

We suppose that F has even degree $2d$, and we homogenize F, setting $u_i = x_i/x_0$. To clear denominators, we must multiply by x_0^{2d}. When we set $y = x_0^d\, w$, we obtain an equation of the form (4.6.12), where f is the homogenization of F. Note that F is square-free if and only if f is square-free.

The structure sheaf $\mathcal{O}_Y$ on a projective double plane Y (4.6.12) can be described by looking above the standard affine open subsets U^i of the projective plane $X = \mathbb{P}^2_x$. Let Y^i be the inverse image of U^i in Y. The regular functions on an open set can be determined from the regular functions on the sets Y^i using the sheaf property of the structure sheaf. The algebra of regular functions on the open set Y^0 is the one defined by the equation (4.6.14).

cubicisdplane **(4.6.15) cubic surfaces and quartic double planes**

We label coordinates of $\mathbb{P}^3$ as $(x, z) = (x_0, x_1, x_2, z)$. Let S be the cubic surface in projective 3-space defined by an irreducible homogeneous cubic polynomal $g(x, z)$, and suppose that $q = (0, 0, 0, 1)$ is a point of S. We'll use π to denote both the projection $(x, z) \rightsquigarrow x$ from $\mathbb{P}^3$ to X and its restriction to S. Since q is a point of S here, the coefficient of z^3 in g is zero. So g is quadratic in z:

projequation (4.6.16)
$$g(z) = a_1 z^2 + a_2 z + a_3,$$

where the coefficients a_i are homogeneous, of degree i in x. The discriminant $a_2^2 - 4a_1a_3$ is a homogeneous polynomial in x, of degree 4.

Let Y be the double plane

quarticdplane (4.6.17)
$$y^2 = a_2{}^2 - 4a_1a_3.$$

and let Δ be its branch locus, the plane curve of degree four defined by the discriminant.

Given a point (x, z) of S, we can pick out a square root y by defining

cubicanddplane (4.6.18)
$$y = 2a_1 z + a_2$$

and the quadratic formula solves for z in terms of y:

quadrformula (4.6.19)
$$z = \frac{y - a_2}{2a_1}$$

These formulas define inverse maps between S and Y that we label as

$$S \xrightarrow{\varphi} Y \quad \text{and} \quad Y \xrightarrow{\psi} S$$

The map φ is undefined at q, and ψ seems to be undefined when $a_1 = 0$. However, when S and q are generic, one can extend the definition of ψ to all of Y.

vdefined **4.6.20. Proposition.** *Let S be a generic cubic surface, and let $S \xrightarrow{\pi} X$ be the projection of S to the projective plane X whose center of projection q is a generic point of S. Let Y be the associated double plane (4.6.17).*

(i) *The map defined by the quadratic formula (4.6.19) extends to a morphism $Y \xrightarrow{\psi} S$.*

(ii) *The line $C : \{a_1 = 0\}$ in X is a bitangent to the quartic branch locus Δ, so it splits in Y. The curves that lie over C are $D_1 : \{a_1 = 0, y = -a_2\}$ and $D_2 : \{a_1 = 0, y = a_2\}$. The image of D_2 in S is the point q, and the image of D_1 in S is the curve E defined by the equations*

$$a_1 = 0, \quad \textit{and} \quad a_2 z + a_3 = 0$$

This is a cubic curve in the plane $\{a_1 = 0\}$, and it has a double point at q.

(iii) *The composition $\psi\varphi$ is defined and is the identity map on S at every point S except q, while $\varphi\psi$ is defined and is the identity map on the complement of D_1 in Y.*

proof. Formula (4.6.17) shows that

$$\frac{y - a_2}{2a_1} = \frac{-2a_3}{y + a_2},$$

so the vectors

definev (4.6.21)
$$(2a_1 x, y - a_2) \quad \text{and} \quad ((y + a_2)x, -2a_3)$$

represent the same point of projective space $\mathbb{P}^3$ wherever they are defined and not zero. The first vector defines $\psi(x, y)$ at points at which a_1 and $y - a_2$ aren't both zero, and the second one defines $\psi(x, y)$ at points at which a_3 and $y + a_2$ aren't both zero. The remaining case is that $a_1 = a_2 = a_3 = 0$. If x is a point of $\mathbb{P}^2$ at which all three of the coefficients a_i vanish, then (x, z) lies on S for every z. Then S contains the line through $(x, 0)$ and $q = (0, 1)$. Since a generic cubic surface contains finitely many lines (**??**), this will not happen when S and q are generic.

When $a_1 = 0$, $g(z) = a_2 z + a_3$ and $y = a_2$. Thus ψ maps D_2 to E, and φ maps the complement of q in E back to D_2. The rest is shown by sorting out the details. □

linesinS **(4.6.22) lines in a cubic surface**

We show now that, as predicted in Chapter **??**, a generic cubic surface contains precisely 27 lines.

Let S be a generic cubic surface in $\mathbb{P}^3$, projected to the plane X from a generic point q of S, as above.

yands (4.6.23)
$$\begin{array}{ccc} Y & \xrightarrow{\psi} & S \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ X & = & X \end{array}$$

As was remarked above, the lines that split in Y will be bitangent lines, provided that the branch locus Δ is generic.

ellmapstoL **4.6.24. Lemma.** *Let ℓ be a line that is contained in S, and let H be the plane in $\mathbb{P}^3$ spanned by ℓ and q.*

(i) *The projection $\mathbb{P}^3 \xrightarrow{\pi} X$ sends H to a line L in X.*

(ii) *L splits in the double plane Y, and is therefore a bitangent to the branch locus Δ.*

(iii) *$Z = H \cap S$ is the union of the line ℓ and a conic.*

proof. **(ii,iii)** The inverse image of L in $\mathbb{P}^3$ is the plane H, and Z is a cubic divisor in H that contains ℓ and q. Since q is generic, it doesn't lie on ℓ. So Z is the union of ℓ and a divisor of degree 2 that contains q. It follows that L isn't the line $C : \{a_1 = 0\}$, whose inverse image is a singular cubic curve.

Next, let W be the inverse image of L in Y. The map ψ carries W surjectively to Z. Since Z isn't irreducible, neither is W. So L splits in the double plane Y, as asserted. Therefore W has two components, and the cubic Z is also the union of two components, one of which is the line ℓ. □

We have seen that an ordinary quartic curve has 28 bitangent lines (**??**), and a generic quartic is ordinary. The lemma below shows that the discriminant locus of a generic cubic will be a generic quartic. Therefore the branch locus Δ of Y has precisely 28 bitangents. One of them is the line $\{a_1 = 0\}$ that was discussed above. It isn't the image of a line in S. The inverse image of any other bitangent consists of a line and a conic. So S contains one line for each bitangent distinct from C. This gives us 27 lines in S.

quarticis-generic

4.6.25. Lemma. *A generic homogeneous quartic $f(x_0, x_1, x_2)$ can be written in the form $a_2^2 - 4a_1a_3$, where a_i is a homogeneous polynomial of degree i.*

proof. We choose for a_1 a linear polynomial such that the line $C : \{a_1 = 0\}$ is a bitangent to the quartic curve $\{f = 0\}$. Then C splits in the double plane, so f is congruent to a square, modulo a_1. Let a_2 be a quadratic polynomial such that $f \equiv a_2^2$ modulo a_1. When we take this polynomial as a_2, we will have $f = a_2^2 + a_1a_3$ for some cubic polynomial a_3. Adjusting the constant puts f into the desired form. □