Chapter 3 PROJECTIVE GEOMETRY

pvariety

3.1 Projective Varieties

As before, the projective space $\mathbb{P}^n$ of dimension n is the set of equivalence classes of nonzero vectors $(x_0, ..., x_n)$, the equivalence relation being that for any nonzero complex number λ,

xlambdax (3.1.1)
$$(x_0, ..., x_n) \sim (\lambda x_0, ..., \lambda x_n).$$

A polynomial $f(x_0, ..., x_n)$ *vanishes* at a point $x = (x_0, ..., x_n)$ of $\mathbb{P}^n$ if $f(\lambda x) = 0$ for every $\lambda \neq 0$, and this happens if and only if each of the homogeneous parts of f vanishes (**??**). So when studying loci of polynomial equations in projective space, we restrict attention to families of polynomials that are homogeneous.

A subset of projective space $\mathbb{P}^n$ is *closed* in the Zariski topology if it is the set of zeros of some homogeneous polynomials. The fact that the polynomial ring $\mathbb{C}[x_0, ..., x_n]$ is noetherian implies that $\mathbb{P}^n$ is a noetherian space. Every closed set is a finite union of irreducible closed sets (2.7.9). The irreducible closed sets are the *closed subvarieties* of $\mathbb{P}^n$.

We studied varieties in $\mathbb{P}^2$ in Chapter **??**. Here are some mote examples of projective varieties.

linsubsp **(3.1.2) linear subspaces**

Let W be a subspace of dimension $r + 1$ of the vector space $V \approx \mathbb{C}^{n+1}$ with basis $(x_0, ..., x_n)$. The points of $\mathbb{P}^n$ that are represented by the nonzero vectors in W form a *linear subspace* L of $\mathbb{P}^n$, of dimension r. If $(w_0, ..., w_r)$ is a basis of W, the linear subspace L corresponds bijectively to $\mathbb{P}^r$, by $c_0 w_0 + \cdots + c_r w_r \leftrightarrow (c_0, ..., c_r)$. For example, the set of points $(x_0, ..., x_r, 0, ..., 0)$ is a linear subspace of dimension r.

quadric-surface **(3.1.3) a quadric surface**

The product $\mathbb{P}^1 \times \mathbb{P}^1$ of projective lines maps bijectively to a quadric in $\mathbb{P}^3_w$. Let coordinates in the two copies of $\mathbb{P}^1$ be (x_0, x_1) and (y_0, y_1), respectively, and let the coordinates in $\mathbb{P}^3$ be w_{ij}, with $0 \leq i, j \leq 1$. The map is defined by $w_{ij} = x_i y_j$, and its image is the quadric Q:

ponepone **(3.1.4)**
$$w_{00} w_{11} = w_{01} w_{10}$$

verone-seemb **(3.1.5) the Veronese embedding of $\mathbb{P}^n$** (Giuseppe Veronese 1854-1917)

Let the coordinates in $\mathbb{P}^n$ be y_i, and let those in $\mathbb{P}^N$ be w_{ij}, $0 \le i,j \le n$. So $N = (n+1)^2-1$. The *Veronese embedding* is the map $\mathbb{P}^n \xrightarrow{u} \mathbb{P}^N$ defined by $w_{ij} = y_i y_j$.

veroneq **3.1.6. Proposition.** *The Veronese embedding is injective, and its image in $\mathbb{P}^N$ is the locus V of the equations*

$$w_{ij}w_{k\ell} = w_{i\ell}w_{kj} \quad \textit{and} \quad w_{ij} = w_{ji}, \quad \textit{for} \quad 0 \le i,j,k,\ell \le n.$$

proof. Let $\mathbb{W}^{ij}$ denote the standard affine subset of $\mathbb{P}^N$ of points at which $w_{ij} \neq 0$, and let $V^{ij} = V \cap \mathbb{W}^{ij}$. The equation $w_{ii}w_{jj} = w_{ij}^2$ shows that $V^{ij} = V^{ii} \cap V^{jj}$. Therefore the open sets V^{jj} cover V. The inverse image of V^{jj} is the standard affine open subset $\mathbb{U}^j$: $\{y_j \neq 0\}$ of $\mathbb{P}^n$.

The equations that define V hold when $w_{ij} = y_i y_j$, so u sends $\mathbb{P}^n$ to V. Suppose given a point w of V. Some coordinate w_{ii}, say w_{00}, must be nonzero. Then the unique solution of the equations with $y_0 = 1$ is $w_{0i} = y_i$, and $w_{00} = 1$. So w is in the image of u. $\square$

When redundancy is removed by setting $w_{10} = w_{01}$, the Veronese embedding sends the projective line $\mathbb{P}^1$ to the conic in $\mathbb{P}^2$ whose equation is $w_{00}w_{11} = w_{01}^2$.

There are higher order Veronese embeddings, defined in the analogous way, using the monomials of some degree $d > 2$.

defprojection **(3.1.7) projection**

The operation π that drops the last coordinate of a point, $\pi(x_0, ..., x_n) = (x_0, ..., x_{n-1})$, is called a projection

projection (3.1.8)
$$\pi : \mathbb{P}^n \to \mathbb{P}^{n-1}$$

It is defined at all points of $\mathbb{P}^n$ except at the *center of projection* $q = (0, ..., 0, 1)$.

To analyze the geometry of the projection, we denote the coordinates in $\mathbb{P}^n$ and $\mathbb{P}^{n-1}$ by $x = x_0, ..., x_n$ and $y = y_0, ..., y_{n-1}$, respectively. The fibre $\pi^{-1}(y)$ over a point y is the set of points (y, x_n) with x_n arbitrary. It is the line ℓ_y through the points $(y, 0)$ and $q = (0, 1)$, with the point q omitted.

Let Γ be the locus in $\mathbb{P}^n \times \mathbb{P}^{n-1}$ defined by the bihomogeneous equations $x_i y_j = x_j y_i$, $0 \le i,j \le n-1$. When we project $\mathbb{P}^n \times \mathbb{P}^{n-1}$ and its subset Γ to $\mathbb{P}^{n-1}$, the fibre Γ_y of Γ over a point y of $\mathbb{P}^{n-1}$ is the set $\ell_y \times y$ of points (x, y) with x in ℓ_y. When we project in the other direction, to $\mathbb{P}^n$, the fibre over a point $(x_0, ..., x_n)$ distinct from q is a single point $(x, \pi(x))$. The fibre over q is the set $q \times \mathbb{P}^{n-1}$, a projective space that maps bijectively to $\mathbb{P}^{n-1}$.

Because the point q of $\mathbb{P}^n$ is replaced by a projective space in Γ, the map $\Gamma \to \mathbb{P}^n$ is called a *blowing up* of the point q of $\mathbb{P}^n$.

figure

segreemb **(3.1.9) the Segre embedding of a product** (Corrado Segre 1863-1924)

The product $\mathbb{P}^m_x \times \mathbb{P}^n_y$ of projective spaces is embedded by the *Segre map* into another projective space $\mathbb{P}^N_w$ with coordinates w_{ij}, $i = 0, ..., m$ and $j = 0, ..., n$. So $N = (m+1)(n+1)-1$ here. The Segre map is defined by

segrecoords (3.1.10)
$$w_{ij} = x_i y_j.$$

It generalizes the map of $\mathbb{P}^1 \times \mathbb{P}^1$ to a quadric that was described in (**3.1.4**).

segreequations **3.1.11. Proposition.** *The Segre map is injective, and its image in $\mathbb{P}^N$ is the locus of solutions of the equations* segreequations (3.1.12)

$$w_{ij}w_{k\ell} - w_{i\ell}w_{kj} = 0.$$

proof. Let's call Π the locus of the equations (3.1.12). When one substitutes (3.1.10) into these equations, they become $x_iy_jx_ky_\ell - x_iy_\ell x_ky_j = 0$, and they are true at every point of $\mathbb{P}^m \times \mathbb{P}^n$. So the image of the Segre map is contained in Π.

Let $\mathbb{W}^{ij}$ denote the standard affine open subset of $\mathbb{P}^N$ at which $w_{ij} \neq 0$. Say that we have a point p of Π at which $w_{00} \neq 0$. We set $w_{00} = 1$. Then $w_{ij} = w_{i0}w_{0j}$, which shows that the coordinates w_{i0} cannot all be zero, and that the same is true of the coordinates w_{0j}. Then the point (x, y) with $x_0 = y_0 = 1$, $x_i = w_{i0}$, and $y_j = w_{0j}$ is the unique point of $\mathbb{P}^m \times \mathbb{P}^n$ whose image is p. This shows that $\mathbb{U}^0 \times \mathbb{U}'^0$ maps bijectively to $\Pi^{00} = \Pi \cap \mathbb{W}^{00}$. □

minorsprime **3.1.13. Note.** The left sides of the equations (3.1.12) are the 2×2 minors (the determinants of the 2×2 submatrices) of the matrix (w_{ij}). A matrix M has rank 1 if and only if all of its 2×2 minors vanish. Thus Π can be identified as the set of rank 1 matrices. It is a fact that for any $r > 0$, the $r \times r$ minors of an $m \times n$-matrix with variable entries generate a prime ideal. This fact isn't easy to prove, and we won't use it. However, Lemma 3.2.5 below shows that the radical of the ideal $\mathcal{P}$ of $\mathbb{C}[w]$ generated by the left sides of (3.1.12) is a prime ideal. So if a homogeneous polynomial $f(w)$ vanishes on Π, then some power of f is in $\mathcal{P}$. □

zartopprod **(3.1.14) The Zariski topology on a product**

Closed subsets of a product $\mathbb{P}^m \times \mathbb{P}^n$ of projective spaces were defined in **??**. A subset of $\mathbb{P}^m \times \mathbb{P}^n$ is *closed* if it is the set of zeros of a system of *bihomogeneous* polynomial equations $f(x, y) = 0$, polynomial equations that are homogeneous in x and in y.

The Zariski topology on a product $X \times Y$ of projective varieties $X \subset \mathbb{P}^m$ and $Y \subset \mathbb{P}^n$ induced from the topology on the product of projective spaces.

It is important to note that the Zariski topology on a product is much finer than the product topology. For example, the proper closed subsets of $\mathbb{P}^1$ are the nonempty finite subsets. In the product topology, the closed subsets of $\mathbb{P}^1 \times \mathbb{P}^1$ are finite unions of points and sets of the form $\mathbb{P}^1 \times p$ and $p \times \mathbb{P}^1$ ('horizontal' and 'vertical' lines). Most loci $\{f = 0\}$, with f bihomogeneous, aren't of this form.

closedinpxp **3.1.15. Proposition.** *A subset of $\mathbb{P}^m \times \mathbb{P}^n$ is closed if and only if its image via the Segre map is closed in $\mathbb{P}^N$. Therefore the map from $\mathbb{P}^m \times \mathbb{P}^n$ to its Segre image is a homeomorphism.*

proof. Let Y be a closed subset of Π, the set of zeros of some homogeneous polynomials $f(w)$ in the variables w_{ij}. Substituting $w_{ij} = x_iy_j$ into any one of the equations $f = 0$, we obtain a bihomogeneous polynomial $\widetilde{f}(x, y) = f(x_iy_j)$, of the same degree as f in x and in y. Thus the zeros of f are images of the zeros of $\widetilde{f}$. So the inverse image of Y in $\mathbb{P}^m \times \mathbb{P}^n$ is closed.

Conversely, let X be a subset of $\mathbb{P}^m \times \mathbb{P}^n$, defined by a bihomogeneous polynomial $g(x, y)$. Say that g has degree r in x and degree s in y. If $r = s$, we may collect the variables in pairs x_iy_j and replace each such pair by w_{ij}, to obtain a homogeneous polynomial in w whose zeros, together with the defining equations (3.1.12), give us the Segre image of X. Suppose that $r \geq s$. Let $k = r - s$. Because the variables y cannot all be zero at any point of $\mathbb{P}^n$, X is also the set of zeros of the system of equations $y_0^kg = y_1^kg = \cdots = y_n^kg = 0$, and these polynomials are bihomogeneous, of the same degree in x and y. □

We note that the mapping property of a product variety:

mapprop **3.1.16. Proposition.** *Let X, Y, Z be varieties. Morphisms $Z \to X \times Y$ correspond bijectively to pairs of morphisms $Z \to X$ and $Z \to Y$.* □

diagonal **(3.1.17) the diagonal**

The *diagonal* X_Δ in a product $X \times X$ is the set of points (x, x).

diagclosed **3.1.18. Proposition.** *Let X be an affine or a projective variety. The diagonal $X_\triangle$ is a closed subvariety of the product $X \times X$.*

proof. We do the case of a projective variety. Let $x'_0, ..., x'_n$ and $x''_0, ..., x''_n$ be coordinates in the two factors of the product $\mathbb{P} \times \mathbb{P}$ $(\mathbb{P} = \mathbb{P}^n)$. The diagonal $\mathbb{P}_\triangle$ in $\mathbb{P} \times \mathbb{P}$ is the closed subvariety defined by the equations $x'_i x''_j = x'_j x''_i$. Next, suppose that X is the closed subvariety of $\mathbb{P}$ defined by a system of homogeneous equations $f(x) = 0$. The diagonal $X_\triangle$ can be identified as the intersection of $X \times X$ with the diagonal $\mathbb{P}_\triangle$, so it is a closed subvariety of $X \times X$. The system of bihomogeneous equations $f(x') = 0$, $f(x'') = 0$ defines the product $X \times X$ as a closed subset of $\mathbb{P}^n \times \mathbb{P}^n$, so $X_\triangle$ can also be described as the closed subvariety of $\mathbb{P} \times \mathbb{P}$ defined by the system of bihomogeneous equations

Xdelta (3.1.19)
$$x'_i x''_j = x'_j x''_i, \quad f(x') = 0, \quad f(x'') = 0$$

□

It is interesting to compare Proposition 3.1.18 with the Hausdorff condition for a topological space. Recall that a topological space X is a *Hausdorff space* if distinct points have disjoint open neighborhoods. A variety with its Zariski topology isn't a Hausdorff space unless it consists of a single point.

The *product topology* on a product $X \times Y$ of topological spaces is the topology whose closed sets are products $C \times D$ of closed sets C and D in X and Y, respectively.

The proof of the next lemma is an exercise in topology:

hausdorff-diagonal **3.1.20. Lemma.** *A topological space X is a Hausdorff space if and only if, when $X \times X$ is given the product topology, the diagonal map $X \xrightarrow{\Delta} X \times X$ defined by $\Delta(x) = (x, x)$ embeds X as a closed subset $X_\triangle$ of $X \times X$.* □

The fact that a variety isn't a Hausdorff space in the Zariski topology doesn't contradict Proposition 3.1.18, because the Zariski topology on the product $X \times X$ is finer than the product topology (see **3.1.14**).

varquasiproj (**3.1.21**) **quasiprojective varieties**

A nonempty (Zariski) open subset X' of a projective variety is called a *quasiprojective variety*. For example, affine varieties and projective varieties are quasiprojective. There are varieties that aren't quasiprojective, but they aren't important, and we work with quasiprojective varieties in these notes. Since the word "quasiprojective" is ugly, and in order to simplify terminology, we will henceforth use the word "variety" to mean "quasiprojective variety", unless the contrary is stated explicitly.

3.2 Homogeneous Ideals

homogen Let $\mathcal{R}$ denote the polynomial ring $\mathbb{C}[x_0, ..., x_n]$.

homogideal **3.2.1. Proposition.** *The following two conditions on an ideal $\mathcal{I}$ of $\mathcal{R}$ are equivalent. An ideal $\mathcal{I}$ is homogeneous if it satisfies either of them.*

- *$\mathcal{I}$ is generated by homogeneous polynomials.*
- *If a polynomial f is in $\mathcal{I}$, then its homogeneous parts are in $\mathcal{I}$.* □

The maximal ideal $\mathcal{M} = (x_0, ..., x_n)$ of $\mathcal{R}$ generated by the variables is special because it has no zeros in projective space. Because of this, $\mathcal{M}$ is sometimes called the *irrelevant ideal*.

nozeros **3.2.2. Proposition.** *Let $\mathcal{I}$ be the ideal of $\mathcal{R}$ generated by homogeneous polynomials $g_1, ..., g_k$. The locus of zeros $g_1 = \cdots = g_k = 0$ in projective space is empty if and only if, either the radical of $\mathcal{I}$ is the irrelevant ideal $\mathcal{M}$, or $\mathcal{I}$ is the unit ideal.*

For the proof, we will want to look at the affine space $\mathbb{A}^{n+1}$ with coordinates $x_0, ..., x_n$, and at the map from the complement of the origin in that affine space to the corresponding point of projective space $\mathbb{P}^n$:

$$\mathbb{A}^{n+1} - \{0\} \longrightarrow \mathbb{P}^n$$

proof. Let Z be the locus of zeros of $\mathcal{I}$ in $\mathbb{P}^n$. If $\operatorname{rad}\mathcal{I} = \mathcal{M}$, then $\mathcal{I}$ contains a power of each variable. Powers of the variables have no common zeros in $\mathbb{P}^n$, so Z is empty.

Suppose that Z is empty, and let W be the locus of zeros of $\mathcal{I}$ in the affine space $\mathbb{A}^{n+1}$ with coordinates $x_0, ..., x_n$. If $a = (a_0, ..., a_n)$ is a point of W other than the origin, and if $\mathcal{I}$ vanishes at a, then the point of $\mathbb{P}^n$ with that coordinate vector is in Z. Since there is no such point, the origin is the only possible point of W. If W is the one point space consisting of the origin, the Strong Nullstellensatz tells us that the radical of $\mathcal{I}$ is $\mathcal{M}$, and if W is empty, $\mathcal{I}$ will be the unit ideal. □

fadicalhomoogeneous **3.2.3. Lemma.** **(i)** *The radical of a homogeneous ideal is homogeneous.*
(ii) *A homogeneous ideal $\mathcal{P}$ that isn't the unit ideal is a prime ideal if and only if, whenever f and g are homogeneous polynomials whose product fg is in $\mathcal{P}$, either f is in $\mathcal{P}$ or g is in $\mathcal{P}$. In other words, a homogeneous ideal is a prime ideal if the condition for a prime ideal is satisfied when the polynomials are homogeneous.*
(iii) *If a homogeneous radical ideal $\mathcal{I}$ is not a prime ideal, there are homogeneous radical ideals $\mathcal{A}, \mathcal{B}$ with $\mathcal{I} < \mathcal{A}$ and $\mathcal{I} < \mathcal{B}$, and that $\mathcal{I} = \mathcal{A} \cap \mathcal{B}$.*

proof. **(i)** Let $\mathcal{I}$ be a homogeneous ideal, and let f be an element of its radical. So f^r is in $\mathcal{I}$ for some r. We write $f = f_0 + \cdots + f_d$ as a sum of its homogeneous parts. The highest degree term of f^r is $(f_d)^r$. Since $\mathcal{I}$ is homogeneous, $(f_d)^r$ is in $\mathcal{I}$ and f_d is in $\operatorname{rad}\mathcal{I}$. Then $f_0 + \cdots + f_{d-1}$ is also in $\operatorname{rad}\mathcal{I}$. By induction on d, all of the homogeneous parts $f_0, ..., f_d$ are in $\operatorname{rad}\mathcal{I}$.

(ii) Suppose that a homogeneous ideal $\mathcal{P}$ satisfies the prime ideal condition for homogeneous polynomials. Let f and g be arbitrary polynomials whose product fg is in $\mathcal{P}$. If f has degree d and g has degree e, the highest degree part of the product fg is $f_d g_e$. It is in the homogeneous ideal $\mathcal{P}$. Therefore one of the factors, f_d or g_e, say f_d, is in $\mathcal{P}$. Let $f' = f - f_d$. Then $f'g$ is in $\mathcal{P}$, and it has lower degree than fg. By induction on the degree of fg, f' or g is in $\mathcal{P}$, and if f' is in $\mathcal{P}$, so is f.

(iii) If a homogeneous ideal $\mathcal{I}$ is not a prime ideal, there are homogeneous polynomials f and g not in $\mathcal{I}$ whose product fg is in $\mathcal{I}$. Let $\mathcal{A} = \mathcal{I} + (f)$ and $\mathcal{B} = \mathcal{I} + (g)$. Then $\mathcal{AB} \subset \mathcal{I} \subset \mathcal{A} \cap \mathcal{B}$. The radicals of $\mathcal{AB}$ and $\mathcal{A} \cap \mathcal{B}$ are equal. Since $\mathcal{I}$ is a radical ideal, so is $\mathcal{A} \cap \mathcal{B}$, and $\mathcal{A} \cap \mathcal{B} = \mathcal{I}$. □

We denote the set of zeros in $\mathbb{P}^n$ of a homogeneous ideal $\mathcal{I}$ by $V(\mathcal{I})$, and the set of zeros of a homogeneous polynomial f by $V(f)$.

homstrnull **3.2.4. Corollary.** *(projective Strong Nullstellensatz)* **(i)** *Let g be a homogeneous polynomial in $x_0, ..., x_n$ that isn't a constant, and let $\mathcal{I}$ be a homogeneous ideal in $\mathbb{C}[x]$. If g vanishes at every point of $V(\mathcal{I})$, then $\mathcal{I}$ contains a power of g.*
(ii) *Let g be a homogeneous polynomial and let f be an irreducible homogeneous polynomial. If $V(f) \subset V(g)$, then f divides g.*

proof. **(i)** Let W be the locus of zeros of $\mathcal{I}$ in the affine space $\mathbb{A}^{n+1}$ with coordinates x. The polynomial g vanishes at every point of W different from the origin. And since g is not a constant, it vanishes at the origin too. So the affine Strong Nullstellensatz 2.6.6 applies. □

The set $V(\mathcal{P})$ of zeros of a homogeneous prime ideal $\mathcal{P}$, not the irrelevant ideal, is called a *projective variety*.

closedirred **3.2.5. Lemma.** *The projective varieties are the irreducible closed subsets of projective space.*

The proof is analogoues to the proof of Lemma 2.7.10. □

homdehom **(3.2.6) homogenization and dehomogenization**

As usual, the standard affine subset $\mathbb{U}^i$ of $\mathbb{P}^n$ is the set of points at which the coordinate x_i is nonzero. The subsets $\mathbb{U}^i$ are n-dimensional affine spaces that form the standard affine cover of $\mathbb{P}^n$. In this section, we discuss the relation between closed subsets of $\mathbb{P}^n$ and closed subsets of the standard affine spaces. We use the standard affine $\mathbb{U}^0$ as an illustration, remembering that the index 0 can be replaced by any other index. We recall that $\mathbb{U}^0 = \operatorname{Spec}\mathbb{C}[u_1, ..., u_n]$, where $u_i = x_i/x_0$.

Let $f(x_0,...,x_n)$ be a homogeneous polynomial. Its *dehomogenization*, with respect to the index 0, is the non-homogeneous polynomial obtained by setting $x_0 = 1$:

dehomog (3.2.7)
$$F(x) = f(1, x_1, ..., x_n).$$

For example, the dehomogenization of $x_0^3 + x_0x_1^2 + x_1x_2^2$ is $1 + x_1^2 + x_1x_2^2$.

The substitution $x_0 = 1$ defines a surjective homomorphism $\mathbb{C}[x_0, ..., x_n] \xrightarrow{\epsilon} \mathbb{C}[x_1, ..., x_n]$, so the process of dehomogenizing is compatible with the algebra structure.

Let $F(x_1,...,x_n)$ be a non-homogeneous polynomial. Its *homogenization* is the homogeneous polynomial $f(x_0,...,x_n)$ obtained as follows: Substitute $x_k = u_k$, where $u_k = x_k/x_0$, then multiply by the smallest power of x_0 required to clear the denominator. Or, multiply each monomial that appears in F by a suitable power of x_0 to bring its degree up to the degree d of F. For example, the homogenization of $1+x_1^2+x_1x_2^2$ is $x_0^3+x_0x_1^2+x_1x_2^2$.

dehomequal **3.2.8. Lemma.** **(i)** *The homogenization of a polynomial F is the unique homogeneous polynomial that isn't divisible by x_0, whose dehomogenization is F.*

(ii) *Homogeneous polynomials f and g whose dehomogenizations are equal differ by a power of x_0.*

(iii) *If two homogeneous polynomials f and g have no common (nonconstant) factor, neither do their dehomogenizations F and G.* □

Thus homogenization and dehomogenization are almost inverse operations:

$$\textit{dehomog} \circ \textit{homog}(F) = F$$

and if x_0 doesn't divide f, then

$$\textit{homog} \circ \textit{dehomog}(f) = f$$

The *dehomogenization* I_0 of a homogeneous ideal $\mathcal{I}$ in $\mathbb{C}[x_0, ..., x_n]$, with respect to the index 0, is the set of dehomogenizations of elements of $\mathcal{I}$, the image of $\mathcal{I}$ via the above homomorphism ϵ. If the dehomogenization G of a homogeneous polynomial g is in I_0, then $x_0^k g$ will be in $\mathcal{I}$ for sufficiently large k.

If a point p of $\mathbb{P}^n$ has coordinates $(1, a_1, ..., a_n)$, its coordinates in the standard affine space $\mathbb{U}^0$ will be $(a_1, ..., a_n)$. So if G is the dehomogenization of a homogeneous polynomial g, then g vanishes at p if and only if G vanishes at the same point, viewed as a point of $\mathbb{U}^0$. Consequently, if $\mathcal{Z}$ is the set of zeros of a homogeneous ideal $\mathcal{I}$ in $\mathbb{P}^n$, the set of zeros in $\mathbb{U}^0$ of the dehomogenized ideal I_0 will be the intersection $\mathcal{Z} \cap \mathbb{U}^0$.

The *homogenization* of an ideal I of $\mathbb{C}[x_1, ..., x_n]$ is the ideal $\mathcal{I}$ *generated by* the homogenizations of the elements of I. It consists of the products $x_0^k f$, where f is the homogenization of an element F of I, and $k \geq 0$. WE allow powers of x_0 here in order to get an ideal.

cIisclosure **3.2.9. Proposition.** *If Z is the set of zeros of an ideal I in the affine space $\mathbb{U}^0$, the set of zeros of its homogenization $\mathcal{I}$ is the closure of Z in $\mathbb{P}^n$.*

proof. Let Y be the zero set of the homogenized ideal $\mathcal{I}$. As we have seen, $Y \cap \mathbb{U}^0 = Z$. Since Y is closed, it contains the closure $\overline{Z}$ of Z in $\mathbb{P}^n$ (see Section 2.7). By the definition of closure, a homogeneous polynomial will vanish on $\overline{Z}$ if and only if it vanishes on Z. We must show that if a homogeneous polynomial f vanishes on Z, then it vanishes on Y. We write $f = x_0^k g$, where x_0 doesn't divide g. Then since x_0 doesn't vanish on Z, g vanishes on Z. It suffices to show that g vanishes on Y. The dehomogenization G of g also vanishes on Z. So for large n, G^n is in I. The homogenization of G^n is g^n. So g^n is in $\mathcal{I}$. Then g^n and g vanish on Y. □

standard-isopen **3.2.10. Corollary.** *The Zariski topology on the standard affine open set $\mathbb{U}^0$ is the topology induced from the Zariski topology on $\mathbb{P}^n$.* □

homogprime **3.2.11. Proposition.** *Let $\mathcal{P}$ be a homogeneous prime ideal in $\mathcal{R} = \mathbb{C}[x_0, ..., x_n]$. Its dehomogenization P with respect to the index 0 is either a prime ideal or the unit ideal. Therefore if X is a variety in $\mathbb{P}^n$, then $X \cap \mathbb{U}^0$ is an affine variety, or else it is empty.*

proof. Suppose that the dehomogenization P of $\mathcal{P}$ isn't the unit ideal. Since the dehomogenization of x_0 is 1, x_0 isn't in $\mathcal{P}$. Let F and G be polynomials whose product FG is in P, and let f, g be homogeneous polynomials whose dehomogenizations are F, G, respectively. The dehomogenization of fg is FG, which is in P. Therefore $x_0^k fg$ is in $\mathcal{P}$ for some k. Since $\mathcal{P}$ is a prime ideal, one of the factors x_0, f, or g is in $\mathcal{P}$. Since x_0 is not in $\mathcal{P}$, f or g is in $\mathcal{P}$, and then F or G is in P. □

closedinaff **3.2.12. Proposition.** **(i)** *If $\{U_i\}$ is a covering of a topological space X by open subsets, a subset X of S is closed if and only if $X \cap U_i$ is closed in U_i for every i.*

(ii) *Let $\{U_i\}$ be the standard affine cover of $\mathbb{P}^n$. A subset X of $\mathbb{P}^n$ is closed if and only if $X \cap U_i$ is closed in U_i for every i.*

This is just topology. □

By the way, when we say that the sets $\{U_i\}$ cover a topological space X, we mean that $X = \bigcup U_i$. We don't allow U_i to contain elements that aren't in X, though that would be customary usage in the english language.

linesinp 3.3 Example: Lines in Projective Three-Space

The *Grassmanian* $G(m, n)$ is a variety that parametrizes subspaces of dimension m of the vector space $\mathbb{C}^n$, or linear spaces of dimension $m-1$ in $\mathbb{P}^{n-1}$. For example, the Grassmanian $G(1, n+1)$ is the projective space $\mathbb{P}^n$. Points of $\mathbb{P}^n$ correspond to one-dimensional subspaces of $\mathbb{C}^{n+1}$. In this section we describe the Grassmanian $G(2, 4)$, which parametrizes lines in $\mathbb{P}^3$ or two-dimensional subspaces of $V = \mathbb{C}^4$. We denote $G(2, 4)$ by $\mathbb{G}$. The point of $\mathbb{G}$ that corresponds to a line ℓ in $\mathbb{P}^3$ will be denoted by $[\ell]$.

One can use row reduction to get insight into the structure of $\mathbb{G}$. A two-dimensional subspace U of $V = \mathbb{C}^4$ will have a basis (u_1, u_2). Let M be the 2×4 matrix whose rows are u_1, u_2. The rows of the matrix obtained from M by row reduction span the same space U, and the row-reduced matrix is uniquely determined by the subspace. Provided that the left hand 2×2 submatrix of M is invertible, the row-reduced matrix will have the form

rowreduced (3.3.1)
$$M' = \begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{pmatrix}$$

So the Grassmanian $\mathbb{G}$ contains as open subset a four-dimensional affine space whose coordinates are the four variable entries of this matrix.

In any 2×4 matrix M with independent rows, some pair of columns will be independent, and those columns can be used in place of the first two in a row reduction. So $\mathbb{G}$ is covered by six four-dimensional affine spaces that we denote by $\mathbb{W}^{ij}$, $1 \le i < j \le 4$, $\mathbb{W}^{ij}$ being the space of 2×4 matrices such that $column_i = e_1$ and $column_j = e_2$. Thus the Grassmanian seems somewhat similar to $\mathbb{P}^4$. It isn't the same space. One difference is that $\mathbb{P}^4$ is covered by five four-dimensional affine spaces.

extalgone **(3.3.2)** **the exterior algebra**

Let V be a complex vector space. The *exterior algebra* $\bigwedge V$ (read 'wedge V') is a ring that contains the complex numbers and is generated by the elements of V, with either of the two equivalent relations

extalg (3.3.3)
$$vv = 0, \quad \text{or} \quad vw = -wv,$$

for all v, w in V.

Suppose that V has dimension four. Let $(v_1, ..., v_4)$ be a basis of V, and let $\bigwedge^i V$ denote the subspace of $\bigwedge V$ spanned by products of length i of elements of V. Then

extbasitwos (3.3.4)

$\bigwedge^0 V = \mathbb{C}$ is a space of dimension 1, with basis $\{1\}$,

$\bigwedge^1 V = V$ is a space of dimension 4, with basis $\{v_1, v_2, v_3, v_4\}$,

$\bigwedge^2 V$ is a space of dimension 6, with basis $\{v_iv_j \mid i<j\} = \{v_1v_2, v_1v_3, v_1v_4, v_2v_3, v_2v_4, v_3v_4\}$,
$\bigwedge^3 V$ is a space of dimension 4, with basis $\{v_iv_jv_k \mid i<j<k\} = \{v_1v_2v_3, v_1v_2v_4, v_1v_3v_4, v_2v_3v_4\}$,
$\bigwedge^4 V$ is a space of dimension 1, with basis $\{v_1v_2v_3v_4\}$,
$\bigwedge^q V = 0$ if $q>4$.

Because it is the direct sum of the subspaces $\bigwedge^i V$ and because multiplication maps $\bigwedge^i V \times \bigwedge^j V$ to $\bigwedge^{i+j} V$, the exterior algebra $\bigwedge V$ is an example of a *noncommutative graded algebra.*

To familiarize yourself with computation in $\bigwedge V$, verify that $v_3v_2v_1 = -v_1v_2v_3$ and that $v_3v_1v_2 = v_1v_2v_3$.

We write an element of $\bigwedge^2 V$ as

wedgetwo (3.3.5)
$$w = \sum_{i<j} a_{ij}v_iv_j.$$

Speaking informally, we regard $\bigwedge^2 V$ as an affine space of dimenion 6, identifying w with the vector whose coordinates are the six coefficients a_{ij}. We also use the symbol w to denote the point of the projective space $\mathbb{P}^5$ with the same coordinates.

An element w of $\bigwedge^2 V$ is *decomposable* if it is a product of two elements of V.

describede-comp **3.3.6. Proposition.** **(i)** *The decomposable elements w of $\bigwedge^2 V$ are those such that $ww=0$, and the relation $ww=0$ is given by the following equation in the coefficients a_{ij}:*

eqgrass (3.3.7)
$$a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23} = 0.$$

(ii) *Suppose given a nonzero decomposable element w of $\bigwedge^2 V$, say $w = u_1u_2$, with u_i in V. The pair (u_1,u_2) is a basis for a two-dimensional subspace U of V.*
(iii) *Let (u_1,u_2) be a basis for a two-dimensional subspace U of V, and let $w = u_1u_2$. Sending $U \rightsquigarrow w$ defines a bijective map from set of two-dimensional subspaces to the locus in $\mathbb{P}^5$ defined by (3.3.7).*

Thus $\mathbb{G} = G(2,4)$ can be represented as a quadric in $\mathbb{P}^5$.

proof. **(i)** If $w = u_1u_2$, then $ww = -u_1^2u_2^2$, which is zero because $u_1^2 = 0$. For the converse, we compute ww when $w = \sum_{i<j} a_{ij}v_iv_j$. The answer is

$$ww = 2\big(a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}\big)v_1v_2v_3v_4.$$

To show that w is decomposable if $ww=0$, it seems simplest to factor w explictly. Since the assertion is trivial when $w=0$, we may suppose that some coefficient of w, say a_{12}, is nonzero. Then if $ww=0$,

factorw (3.3.8)
$$w \;=\; \frac{1}{a_{12}}\big(a_{12}v_2 + a_{13}v_3 + a_{14}v_4\big)\big(-a_{12}v_1 + a_{23}v_3 + a_{24}v_4\big).$$

(ii) If an element w of $\bigwedge^2 V$ is decomposable, say $w = u_1u_2$, and if w is nonzero, then u_1 and u_2 must be independent. They span a two-dimensional subspace. Conversely, let (u_1,u_2) be a basis for a subspace U of dimension 2, and let $w = u_1u_2$. Changing the basis by a 2×2 invertible matrix P multiplies the product w by the determinant of P. So the point w of $\mathbb{P}^5$ that corresponds to U is determined uniquely.

Finally, let (u_1,u_2) and (u_1',u_2') be bases for distinct two-dimensional subspaces U and U'. At least three of the vectors $\{u_1,u_2,u_1',u_2'\}$ will be independent, and therefore $u_1u_2 \neq \lambda u_1'u_2'$ (see 3.3.4). $\square$

We will use the *algebraic dimension* of a variety here, though this concept won't be studied until Chapter **??**. We refer to the algebraic dimension simply as the *dimension*. By definition, the dimension of a variety X is defined to be the length d of the longest chain $C_0 > C_1 > \cdots > C_d$ of closed subvarieties of X. (In a longest chain, C_0 will be the whole space X, and C_d will be a point.) The topological dimension of X, its dimension in the classical topology, is always twice the algebraic dimension. The Grassmanian $\mathbb{G}$, for example, has dimension 4.

pinlclosed **3.3.9. Proposition.** *Let $\mathbb{P}^3$ be the projective space associated to the vector space V. In the product $\mathbb{P}^3\times\mathbb{G}$, the locus $\Gamma = \{p, [\ell] \,|\, p \in \ell\}$ of pairs such that p lies on ℓ is a closed subset of dimension* 5.

proof. Say that ℓ is the line in $\mathbb{P}^3$ that corresponds to the subspace U with basis (u_1, u_2), and that p is represented by the vector x of V. Let $w = u_1u_2$. Then $p \in \ell$ means $x \in U$, which is true if and only if (x, u_1, u_2) is a dependent set, or if $xw = 0$. So Γ is the closed subset of $\mathbb{P}^3\times\mathbb{P}^5$ defined by the bihomogeneous equations $ww = 0$ and $xw = 0$. The fibre of Γ over the point $[\ell]$ of $\mathbb{G}$ is the set of points of ℓ. Thus Γ can be viewed as a four-dimensional family of lines, parametrized by $\mathbb{G}$. Its dimension is $4 + 1 = 5$. □

linesina-surface **(3.3.10) lines on a surface**

Suppose given a surface S in $\mathbb{P}^3$. We ask: Does S contain a line? One surface that contains lines is the quadric Q with equation $w_1w_2 = w_0w_3$, the image of the Segre map $\mathbb{P}^1\times\mathbb{P}^1 \to \mathbb{P}^3_w$ (**3.1.3**). It contains two families of lines, corresponding to the two "rulings" $p\times\mathbb{P}^1$ and $\mathbb{P}^1\times q$ of $\mathbb{P}^1\times\mathbb{P}^1$. There exist surfaces of arbitrary degree that contain lines, but we will see that a generic surface of degree four or more contains no line.

We use coordinates x_i with $i = 1, 2, 3, 4$ for $\mathbb{P}^3$ here. There are $\binom{d+3}{3}$ monomials of degree d in four variables, so surfaces of degree d in $\mathbb{P}^3$ are parametrized by a projective space of dimension $N = \binom{d+3}{3} - 1$. Let $\mathbb{S}$ denote that projective space. The coordinates of the point $[S]$ of $\mathbb{S}$ that corresponds to a surface S are the coefficients of the monomials in its defining polynomial f. Speaking infomally, we say that a point of $\mathbb{S}$ "is" a surface of degree d in $\mathbb{P}^3$. (If f is reducible, its zero locus isn't a variety. Let's not worry about this.)

Consider the line ℓ_0 through the points $e_1 = (1, 0, 0, 0)$ and $e_2 = (0, 1, 0, 0)$. A surface $S : \{f = 0\}$ will contain ℓ_0 if and only if $f(x_1, x_2, 0, 0) = 0$ for all x_1, x_2. Substituting $x_3 = x_4 = 0$ into f leaves us with a polynomial in two variables:

scontainslzero (3.3.11)
$$f(x_1, x_2, 0, 0) = c_0x_1^d + c_1x_1^{d-1}x_2 + \cdots + c_dx_2^d,$$

where c_i are some of the coefficients of f. In order for this polynomial to be identically zero, all of its coefficients must be zero. So the surfaces that contain ℓ_0 correspond to the points of the linear subspace $\mathbb{L}_0$ of $\mathbb{S}$ defined by $c_0 = \cdots = c_d = 0$. This is a satisfactory answer to the question of which surfaces contain ℓ_0, and we can use it to make a guess about lines in a generic surface of degree d.

scontainslclosed **3.3.12. Lemma.** *In $\mathbb{G}\times\mathbb{S}$, the set Ξ of pairs $[\ell], [S]$ such that $\ell \subset S$ is a closed subset.*

proof. Let $\mathbb{W}^{ij}$, $1 \le i < j \le 4$ denote the six standard affine spaces that cover the Grassmanian. It suffices to show that the intersection $\Xi^{ij} = \Xi \cap (\mathbb{W}^{ij}\times\mathbb{S})$ is closed in $\mathbb{W}^{ij}\times\mathbb{S}$ (3.2.12). We inspect the case $i, j = 1, 2$.

A line ℓ such that $[\ell]$ is in $\mathbb{W}^{12}$ has a basis of the form u_1, u_2, where $u_1 = (1, 0, a_2, a_3)$ and $u_2 = (0, 1, b_2, b_3)$ for some uniquely determined a_i, b_i, and ℓ is the line $\{ru_1 + su_2\}$. Let $f(x_1, x_2, x_3, x_4)$ be the defining polynomial of a surface S. Then ℓ is contained in S if and only if $\widetilde{f}(r, s) = f(r, s, ra_2 + sb_2, ra_3 + sb_3)$ is zero for all r and s, and $\widetilde{f}$ is a homogeneous polynomial of degree d in r, s. If we write $\widetilde{f} = z_0r^d + z_1r^{d-1}s + \cdots + z_ds^d$, the coefficients z_j will be polynomials in a, b and in the coefficients of f, and they will be homogeneous and linear in the coefficients of f. The zero locus of these coefficients is the closed set Ξ^{ij}. □

The set of surfaces that contain our special line ℓ_0 is the linear subspace $\mathbb{L}_0$ of $\mathbb{S}$ defined by the vanishing of $d+1$ coordinates. So the dimension of $\mathbb{L}_0$ is $N-d-1$. The line ℓ_0 can be carried to any other line ℓ_1 by a linear map $\mathbb{P}^3 \to \mathbb{P}^3$, so the sufaces that contain ℓ_1 also form a linear subspace of dimension $N-d-1$ in $\mathbb{S}$, and Ξ is the union of such linear spaces. The dimension of the Grassmanian $\mathbb{G}$ is 4. Therefore the dimension of Ξ is

dimspace-lines (3.3.13)
$$\dim\Xi \;=\; (N-d-1) + 4 \;=\; N-d+3 \;=\; \dim\mathbb{S} - d + 3.$$

We project the product $\mathbb{G}\times\mathbb{S}$ and its subvariety Ξ to $\mathbb{S}$. The fibre of Ξ over a point $[S]$ is the set of pairs $[\ell], [S]$ such that ℓ is contained in S. It can be identified with the set of lines in S.

When the degree d is 1, $\dim\Xi = \dim\mathbb{S} + 2$. Every fibre of Ξ over $\mathbb{S}$ will have dimension at least 2. In fact, every fibre has dimension equal to 2. Surfaces of degree 1 are planes, and the lines in a plane form a two-dimensional family.

When $d = 2$, $\dim \Xi = \dim \mathbb{S} + 1$. We can expect that most fibres of Ξ over $\mathbb{S}$ will have dimension 1. This is true: A smooth quadric contains two one-dimensional families of lines. (All smooth quadrics are equivalent with the quadric (**3.1.4**).) But if a quadratic polynomial $f(x_1, x_2, x_3, x_4)$ is the product of linear polynomials, its locus of zeros will be a union of planes. It will contain two-dimensional families of lines. Some fibres of Ξ over $\mathbb{S}$ have dimension 2.

When $d \geq 4$, $\dim \Xi < \dim \mathbb{S}$. The projection $\Xi \to \mathbb{S}$ cannot be surjective. Most surfaces of degree 4 or greater contain no lines.

The most interesting case is that $d = 3$. Then $\dim \Xi = \dim \mathbb{S}$. Most fibres will have dimension zero. They will be finite sets. In fact, a generic cubic surface contains 27 lines. We have to wait to see why the number is precisely 27 (see (**??**)).

Our conclusions are intuitively plausible, but to be sure about them, we will need to study dimension carefully.

regfunction

3.4 Regular Functions

Let $\mathbb{U}^i$ denote the standard affine open subset of projective space $\mathbb{P}^n_x$. We recall that $\mathbb{U}^j$ is an affine space, whose coordinate algebra is the polynomial ring $R_j = \mathbb{C}[u_{0j}, u_{1j}, ..., u_{nj}]$, $u_{ij} = x_i/x_j$ and $ujj = 1$. The intersection $\mathbb{U}^{jk} = \mathbb{U}^j \cap \mathbb{U}^k$ is also an affine variety whose coordinate ring is either of the isomorphic localizations $R_j[u_{kj}^{-1}] \approx R_k[u_{jk}^{-1}]$.

Let X be a projective variety, a closed subvariety of $\mathbb{P}^n$, and let X^j denote the closed subset $X \cap \mathbb{U}^j$ of $\mathbb{U}^j$. If X^j is empty, we ignore it. If X^j isn't empty, it will be an affine variety that is also an open subset of X. The coordinate ring A_j of X^j is a quotient of the coordinate ring R_j of $\mathbb{U}^j$. Specifically, if X is the locus of zeros of some homogeneous polynomials $f(x)$, and if $F(u)$ denotes the dehomogenization of $f(x)$ with respect to the index j, then $A_j = R_j/(F)$. If X^j happens to be empty, we ignore it.

If both X^j and X^k are nonempty, they will be dense in X, and therefore $X^{jk} = X^j \cap X^k = X \cap \mathbb{U}^{jk}$ will also be dense in X. In that case, X^{ij} is the affine variety whose coordinate ring A_{jk} is a localization of A_j and of A_k. If s is the residue of u_{jk} in A_j, then $A_{jk} = A_j[s^{-1}]$.

fnfldindepofi **3.4.1. Lemma.** *Let X^j and A_j be as above. If X^j and X^k are both nonempty, the fields of fractions of A_j and A_k are canonically isomorphic.* □

The *function field* K of a projective variety X is the field of fractions of any of the algebras A_j such that $X^j \neq \emptyset$. And, if X' is a nonempty open subset of a projective variety X, a quasiprojective variety, the function field of X' is the function field of its closure X. A *rational function* on X is an element of the function field K.

Let p be a point of X^j. A rational function α on X is *regular at* p if it can be written as a fraction a/b where a, b are in A_j and b isn't zero at p.

regindep **3.4.2. Lemma.** *The regularity of a rational function α at a point p doesn't depend on the index j such that $p \in X^j$.* □

The points p at which a rational function α is regular form an open subset U of X. Namely, when α is written as a fraction a/b with a and b in A_j, it will be regular at all points of the open subset of X^j at which b isn't zero. The set U will be a union of such open sets.

If a rational function α is regular at every point of a variety X, it is called a *regular function on* X. There may not be many regular functions. It is a fact that on a projective variety, the regular functions are the constants. We learn how to prove this later (see **??**).

If X is an affine variety $\operatorname{Spec} A$, we may define regularity by embedding X as a closed subset of the standard affine open subset $\mathbb{U}^0$ of $\mathbb{P}^n$. Then the elements of A are regular functions on X. The next proposition shows that there are no other regular functions on X.

deloca **3.4.3. Proposition.** *Let X be the affine variety $\operatorname{Spec} A$. The regular functions on X, as defined above, are the elements of A.*

proof. Let α be a regular function on X. Then for every point p of X, there is a simple localization $X_s = \operatorname{Spec} A_s$ such that α is an element of A_s. A finite set of these localizations, say $X_{s_1}, \ldots, X_{s_k}$, will cover X (Corollary **??**). Then the elements s_i have no common zeros on X, and therefore they generate the unit ideal of A. The next lemma completes the proof.

sheafaxiomA **3.4.4. Lemma.** *Let K be the field of fractions of a domain A, and let $s_1, ..., s_k$ be nonzero elements of A that generate the unit ideal, An element α of K that is in A_{s_i} for every i is in A.*

proof. Since α is in A_{s_i}, we can write it as a fraction $\alpha = s_i^{-n} b_i$, with b_i in A. Then $s_i^n \alpha = b_i$. We can use the same large exponent n for each i. Since the elements s_i generate the unit ideal of A, so do the powers s_i^n. Say that $\sum s_i^n r_i = 1$, with r_i in A. Then $\alpha = \sum s_i^n r_i \alpha = \sum r_i b_i$, and $r_i b_i$ is in A. □

ratfnspspace **(3.4.5) rational functions on projective space**

Let $\mathcal{R}$ denote the polynomial ring $\mathbb{C}[x_0, ..., x_n]$, and let f be a homogeneous polynomial. Though it makes sense to say that f vanishes at a point of $\mathbb{P}^n$, it doesn't define a function on $\mathbb{P}^n$. The reason is that, if f has degree d, then $f(\lambda x) = \lambda^d f(x)$. On the other hand, a fraction g/h of homogeneous polynomials of the same degree d defines a function wherever $h \neq 0$, because $g(\lambda x)/h(\lambda x) = (\lambda^d g(x))/(\lambda^d h(x)) = g(x)/h(x)$. Thus rational functions on $\mathbb{P}^n$ can be represented as fractions g/h of homogeneous polynomials of the same degree.

homogfractsfn-fld **3.4.6. Lemma.** **(i)** *The rational function defined by a fraction g/h of relatively prime homogeneous polynomials of the same degree is regular at a point x of $\mathbb{P}^n$ if and only if $h(x) \neq 0$.*
(ii) *Let h be a nonconstant, homogeneous polynomial of positive degree d, let Z be the set of zeros of h in $\mathbb{P}^n$, and let U be the complement of Z in $\mathbb{P}^n$. The nonzero rational functions that are regular on U are those of the form g/h^k, where g is a homogeneous polynomial of degree dk and $k \geq 0$.* □

proof. **(i)** Say that p is a point of the standard affine open $\mathbb{U}^0$. We set $x_0 = 1$. Let the dehomogenizations of g and h be $\widetilde{g}(x) = g(1, x_1, ..., x_n)$ and $\widetilde{h}(x) = h(1, x_1, ..., x_n)$. The polynomials $\widetilde{g}, \widetilde{h}$ have no common factor, and $\alpha = \widetilde{g}/\widetilde{h}$. So α is regular at p if and only if $\widetilde{h}(x) \neq 0$.

(ii) Let $\alpha = g_1/h_1$ be a regular function on U, with g_1, h_1 relatively prime. So h_1 doesn't vanish on U. By the Strong Nullstellensatz, h_1 divides a power h^k of h, say $h^k = f h_1$. Then $g_1/h_1 = f g_1/f h_1 = f g_1/h^k$ □

indeterminate **Note.** Let g and h be relatively prime homogeneous polynomials of degree d, and let Y and Z be their zero sets in $\mathbb{P}$, respectively. The rational function $\alpha = g/h$ will tend to infinity as one approaches a point of Z that isn't also a point of Y. At intersections of Y and Z, α is indeterminate. For example, let the coordinates in $\mathbb{P}^2$ be x, y, z. The rational function y/x is indeterminate at the point $p = (0, 0, 1)$. If one approaches p along the line $\ell_\lambda : \{y = \lambda x\}$, the limit is λ. □

3.5 Morphisms and Isomorphisms

sectmorph

Morphisms of projective varieties are defined in terms of regular functions. They cannot be defined directly in terms of the projective coordinates, as the next example illustrates.

lineconic **3.5.1. Example.** Let coordinates in $\mathbb{P}^1$ and $\mathbb{P}^2$ be x_0, x_1 and w_{00}, w_{01}, w_{11}, respectively. We define a map $\mathbb{P}^1 \xrightarrow{\psi} \mathbb{P}^2$ by $\psi(x_0, x_1) = (x_0^2, x_0 x_1, x_1^2) = (w_{00}, w_{01}, w_{11})$. This is the map that is obtained from the Veronese embedding (**3.1.5**) by eliminating the redundant coordinate w_{10}. Its image is the conic C:

$$w_{00} w_{11} = w_{01}^2.$$

The map ψ ought to be an isomorphism from $\mathbb{P}^1$ to C. But an isomorphism has an inverse, and because the coordinates of $\psi(x)$ are quadratic polynomials in x, the inverse isn't given globally by polynomials in w.

To define the inverse morphism, let C^i denote the open subset $w_{ii} \neq 0$ of C. The The curve C is covered by the two opens sets $C^0 : \{w_{00} \neq 0\}$ and $C^1 : \{w_{11} \neq 0\}$. We define $C^0 \xrightarrow{\theta^0} \mathbb{P}^1$ by $\theta^0(w) = (w_{00}, w_{01})$ and $C^1 \xrightarrow{\theta^1} \mathbb{P}^1$ by $\theta^1(w) = (w_{00}, w_{01})$ and $C^1 \xrightarrow{\theta^0} \mathbb{P}^1$ by $\theta^1(w) = (w_{01}, w_{11})$. Then

$$\psi\theta^0(w) = \psi(w_{00}, w_{01}) = (w_{00}^2, w_{00}w_{01}, w_{01}^2) = (w_{00}^2, w_{00}w_{01}, w_{00}w_{11}) \sim (w_{00}, w_{01}, w_{11}) = w$$

Similarly, $\psi\theta^1(w) = w$. The maps θ^0 and θ^1 piece together to define the inverse of ψ. □

If $Y \xrightarrow{u} X$ is a map of sets and if α is a function on X, the composition αu is a function on Y called the *pullback* of α. The pullback is often denoted by $u^*\alpha$, the asterisk in the superscript position indicating that the pullback goes in the opposite direction from the map u: $\mathrm{Funct}(Y) \xleftarrow{u^*} \mathrm{Funct}(X)$.

$$\begin{array}{ccc} Y & \xrightarrow{u} & X \\ {\scriptstyle u^*\alpha}\downarrow & & \downarrow{\scriptstyle \alpha} \\ \mathbb{C} & = & \mathbb{C} \end{array}$$

defmorph **3.5.2. Definition.** Let X and Y be varieties. A *morphism* $Y \xrightarrow{u} X$ is a continuous map with this property: Let q be a point of Y. If p is the image of q in X, and if f is a rational function on X that is regular at p, its pullback u^*f is a regular function at q. An *isomorphism* of varieties $Y \xrightarrow{u} X$ is a bijective morphism whose inverse function is also a morphism.

The Veronese map (**3.1.5**) defines an isomorphism onto its image, the locus of the equations (3.1.6).

All parts of the next lemma are easy to verify.

firstprop-morph **3.5.3. Lemma.** **(i)** *The following are morphisms:*

- *the composition fg of two morphisms $Z \xrightarrow{g} Y \xrightarrow{f} X$,*
- *the inclusion of a nonempty open subset U into X,*
- *the inclusion of a closed subvariety Y into X.*

(ii) *Let $Y \xrightarrow{f} X$ be a morphism whose image lies in an open or a closed subvariety Z of X. The restricted map $Y \to Z$ is a morphism.*

(iii) *Let $Y \xrightarrow{f} X$ be a continuous map, let Y_i be open subvarieties of Y that cover Y, $Y = \bigcup Y_i$. If the restrictions $Y_i \xrightarrow{g} X$ of f are morphisms, so is f.* □

morphtoline **3.5.4. Proposition.** *Morphisms from an arbitrary variety Y to the affine line $\mathbb{A}^1 = \mathrm{Spec}\,\mathbb{C}[x]$ correspond bijectively to regular functions on Y.*

proof. Let $Y \xrightarrow{u} \mathbb{A}^1$ be a morphism. Since the coordinate function x on $\mathbb{A}^1$ is regular, its pullback $\alpha = u^*x$ is a regular function on Y. If a point q of Y has image a in $\mathbb{A}^1$, i.e., if $u(q)$ is the point $x = a$, then $\alpha(y) = a$.

Conversely, let α be a regular function on Y. We define a map $Y \xrightarrow{u} \mathbb{A}^1$ as follows: Given a point q of Y, we define $u(q)$ to be the point $x = a$, where $a = \alpha(q)$. Then $u^*x(q) = x(u(q)) = a$. So $u^*x = \alpha$. Then the pullback $g(\alpha)$ of a polynomial $g(x)$ will be regular too. The regular functions on $\mathbb{A}^1$ are the polynomial functions, so the pullback of every regular function on $\mathbb{A}^1$ is regular, and u is a morphism. □

Xmaps **3.5.5. Corollary.** **(i)** *Let Y be a variety, and let A be a finite-type algebra whose elements are regular functions on Y. There is a canonical morphism $Y \to \mathrm{Spec}\,A$.*
(ii) *Let Y and X be affine varieties $\mathrm{Spec}\,B$ and $\mathrm{Spec}\,A$, respectively. Morphisms $Y \xrightarrow{u} X$, as defined in (3.5.2), correspond bijectively to algebra homomorphisms $A \xrightarrow{\varphi} B$.*

Thus the definition of morphism given here agrees with the one given for affine varieties in Chapter 2.

proof. **(i)** We choose a finite set $\alpha_1, ..., \alpha_k$ of algebra generators for A, so that $A \approx \mathbb{C}[x_1, ..., x_k]/P$ for some prime ideal P. The generators define morphisms $Y \xrightarrow{u_i} \mathbb{A}^1$ as in the previous proposition, and therefore a morphism $Y \xrightarrow{v} \mathbb{A}^k$. Since the functions in Y defined by elements of P are zero, the image of v is contained in the locus $V(P)$, which is $\mathrm{Spec}\,A$.
Part **(ii)** follows from part **(i)**. □

morphtopn **(3.5.6) morphisms to projective space**

We analyze morphisms from an arbitrary variety Y to projective space.

Let k be a field that contains the complex numbers. Points of projective space with values in k are defined in the same way as ordinary points, points with values in $\mathbb{C}$, as equivalence classes of nonzero vectors $(\alpha_0,...,\alpha_n)$ with α_i in k^n. The equivalence relation is that $(\alpha_0,...,\alpha_n) \sim (\lambda\alpha_0,...,\lambda\alpha_n)$ for any nonzero element λ of k. As with points with values in $\mathbb{C}$, one can represent the point (α) in a unique way, with α_0 normalized to 1, provided that α_0 isn't zero. If X is a projective variety, say $X \subset \mathbb{P}^n$, a point of X with values in k is a point of $\mathbb{P}^n$ with values in k, that solves the homogeneous polynomial equations that define X, i.e., that "lies on X".

maptopn **3.5.7. Proposition** *Let K be the function field of a variety X.*

(i) *A morphism $X \to \mathbb{P}^n$ determines a point $(z_0,...,z_n)$ of $\mathbb{P}^n$ with values in K.*

(ii) *A vector $(z_0,...,z_n)$ with $z_i \in K$ determines a unique morphism $X \xrightarrow{f} \mathbb{P}^n$ if and only if for every point $q \in X$, there is an index i such that $z_i \neq 0$, and for $j = 0,...,n$, the rational functions $g_{ij} = z_j/z_i$, with $g_{ii} = 1$, are regular at q.*

proof. **(ii)** Such a vector (z) does determine a map to $X@>f>>\mathbb{P}^n$, by $f(q) = g_{ij}(q)$. Say that $p = f(q)$ is in the standard affine $\mathbb{U}^i$. Then p is the point $u_{ij}(p) = g_{ij}(q)$. If α is a rational function on $\mathbb{P}^n$ that is regular at p, we can write $\alpha = a/s$ where a, s are polynomials in u_{ij} and $s(p) \neq 0$. Then $\alpha f = a(g)/s(g)$. Since $s(f(p)) = s(p) \neq 0$, αf is regular at q. □

maptopn **3.5.8. Proposition** *Let K be the function field of a variety X.*

(i) *A morphism $X \to \mathbb{P}^n$ determines a point $(z_0,...,z_n)$ of $\mathbb{P}^n$ with values in K.*

(ii) *A vector $(z_0,...,z_n)$ with $z_i \in K$ determines a unique morphism $X \to \mathbb{P}^n$ if and only if for every point $q \in X$, there is an index i such that $z_i \neq 0$, the rational functions $g_{ij} = z_j/z_i$ are regular at q for all $j = 0,...,n$, and not all g_{ij} are zero at q.*

proof. **(ii)** Such a vector (z) does determine a map to $X \xrightarrow{f} \mathbb{P}^n$, by $f(q) = g_{ij}(q)$. Say that $p = f(q)$ is in the standard affine $\mathbb{U}^j$ with coordinates $u_{ij} = x_j/x_i$. Then p is the point $u_{ij}(p) = g_{ij}(q)$. If α is a rational function on $\mathbb{P}^n$ that is regular at p, we can write $\alpha = a/s$ where a, s are polynomials in u_{ij} and $s(p) \neq 0$. Then $\alpha f = a(g)/s(g)$. Since $s(f(p)) = s(p) \neq 0$, αf is regular at q. □

morphisom **3.5.9. Corollary.** *Let $Y \xrightarrow{f} X$ be a morphism, let $\{X_i\}$ be an open covering of X, and let $Y_i = f^{-1}X_i$. If the morphism f restricts to an isomorphism $Y_i \xrightarrow{f_i} X_i$ for each i, then f is an isomorphism.*

proof. Say that Y is a subvariety of $\mathbb{P}^r$. Let g_i be the inverse morphism of f_i. We may regard g_i as a morphism $X_i \to \mathbb{P}^r$ These morphisms agree on the intersections $X_i \cap X_j$. Therefore they are defined by the same point $(z_0,...,z_r)$ with values in the function field K of X. Since the open sets X_i cover X, the condition of Proposition 3.5.8 is satisfied, so (z) defines a morphism $X \xrightarrow{g} \mathbb{P}^r$ that restricts to g_i on X_i. Therefore g is an isomorphism. □

affopens **(3.5.10) affine open subvarieties**

Since we now have the concept of an isomorphism, we can define affine subvarieties of a variety. An open subvariety U of a variety X is *affine* if it is isomorphic to an affine variety $\operatorname{Spec} A$.

Suppose that the ring A of all rational functions that are regular on an open set U is a finite-type domain. Then $\operatorname{Spec} A$ will be an affine variety, and we will have a morphism $U \to \operatorname{Spec} A$ (see Corollary 3.5.5). Then U is an affine open subvariety if this morphism is an isomorphism. Speaking informally, we will often identify U with $\operatorname{Spec} A$.

Most open subsets in a variety X aren't affine, and it is often difficult to decide whether a given open set is affine or not. What we know mainly is that the affine open sets form a basis for the topology on a variety X. The next theorem gives us another bit of information.

comphyper ## Example: complement of hypersurface in $\mathbb{P}^n$ is affine ##

intersectaffine **3.5.11. Theorem** *If U, V are affine open subsets of a variety X, their intersection $U \cap V$ is an affine open subset.*

openopen **3.5.12. Lemma. (i)** *Let $U \subset V \subset X$ be inclusions of affine varieties, with U open in V and V open in X. If U is a localization of X, it is also a localization of V. If U is a localization of V and V is a localization of X, then U is a localization of X.*

(ii) *Let U and V be affine open subsets of a variety X, and let p be a point of $U \cap V$. There is an affine open subset Z of X that contains p and that is a localization, both of U and of V.*

proof. **(i)** Say that V is the localization X_s of X, the set of points of X at which $s \neq 0$. Then V is also the set of points of U at which $s \neq 0$, so $V = U_s$. Next, suppose that $U = X_s$ and $V = U_t$. So $U = \operatorname{Spec} A_s$ and t is an element of that ring, say $t = s^{-k}x$. Then $V = U_t = U_x = X_{sx}$.

(ii) Because localizations form a basis, there is a localization U_s of U that contains p and is contained in $U \cap V$, and there is a localization V_t of V that contains p and is contained in U_s: $V_t \subset U_s \subset V$. By **(i)**, V_t is a localization of U. □

proof of Theorem 3.5.11. We may assume that X is closed in a projective space $\mathbb{P}^n$. Say that $U = \operatorname{Spec} A$, that $V = \operatorname{Spec} B$, and that $A = \mathbb{C}[\alpha_1, ..., \alpha_r]$ and $B = \mathbb{C}[\beta_1, ..., \beta_s]$, where α_i and β_j are rational functions on X. Let $R = \mathbb{C}[\alpha, \beta]$ denote the algebra generated by the two algebras A and B, and let $W = \operatorname{Spec} R$. The plan is to show that $U \cap V = W$. We note that the inclusions of coordinate algebras give us morphisms $W \to U$ and $W \to V$, and of course we have the inclusions $U \to X$ and $V \to X$.

Let $(z_0, ..., z_n)$ be the point of $\mathbb{P}^n$ with values in the function field K of X that defines the projective embedding of X. Since K is the function field of U, V, W as well as of X, this point defines unique morphisms from U, V, W, and X to $\mathbb{P}^n$. Since these morphisms are unique, they are compatible with the morphisms between these varieties mentioned above.

$$\begin{array}{ccc} W & \longrightarrow & U \\ \big\downarrow & & \big\downarrow \\ V & \longrightarrow & X \end{array}$$

Therefore the image of W is contained in $U \cap V$. So the point (z) defines a morphism $W \xrightarrow{f} U \cap V$ that we plan to show is an isomorphism.

Let p be a point of $U \cap V$. Lemma 3.5.12 shows that there is an affine open subset Z of $U \cap V$ that is a localization of U and of V, and that contains p. Let S be the coordinate ring of Z. Then $S = A_s = B_t$ for some $s \in A$ and $t \in B$. If P and Q are two algebras, let's denote the algebra they generate by $[P, Q]$. So $R = [A, B]$. With this notation,

$$R_s = [A, B]_s = [A_s, B] = [B_t, B] = B_t = S$$

So S is a localization of R. Then $\operatorname{Spec} S = Z$ identifies with an open subset of W which maps bijectively to the subset Z of $U \cap V$. Corollary 3.5.9 shows that the map $W \to U \cap V$ is an isomorphism. □

mappropprod **(3.5.13)** **the mapping property of a product variety**

Let X, Y and Z be sets, and let $X{\times}Y \xrightarrow{\pi_X} X$ and $X{\times}Y \xrightarrow{\pi_Y} Y$ be the projection maps. Maps $Z \xrightarrow{u} X{\times}Y$ correspond bijectively to pairs f, g of maps $Z \xrightarrow{f} X$ and $Z \xrightarrow{g} Y$, as follows:

Given f and g, $u(z) = (f(z), g(z))$.
Given u, $f(z) = \pi_X \circ u(z)$ and $g(z) = \pi_Y \circ u(z)$.

We show here that products of varieties have the analogous property.

Products of projective varieties were discussed in Section 3.1. If X and Y are quasiprojective varieties that are open subsets of projective varieties $\overline{X}$ and $\overline{Y}$ respectively, the product $X \times Y$ is an open subset of $\overline{X} \times \overline{Y}$, so it is a quasiprojective variety too.

mappr **3.5.14. Theorem.** *Let X, Y and Z be varieties.*
(i) *The projections from the product $X \times Y$ to X and Y are morphisms*

$$\begin{array}{ccc} X \times Y & \xrightarrow{\pi_2} & Y \\ {\scriptstyle \pi_1}\downarrow & & \\ X & & \end{array}$$

(ii) *Using the rules given above, morphisms $Z \xrightarrow{u} X \times Y$ correspond bijectively to pairs f, g of morphisms $Z \xrightarrow{f} X$ and $Z \xrightarrow{g} Y$.*

proof. Lemma 3.5.3 **(iii)** shows that it suffices to prove **(ii)** when Z is affine.

We check **(i)** and **(ii)** first when $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ are affine varieties. Say that $A = \mathbb{C}[u_1, ..., u_m]/(f)$ and $B = \mathbb{C}[v_1, ..., v_n]/(g)$, where f and g are families of polynomials. Then the product $X{\times}Y$ can be identified as the locus in affine space $\mathbb{A}^{m+n}$ with coordinates $u_1, ..., u_m, v_1, ..., v_n$ that is defined by the system of equations $f(u) = 0$ and $g(v) = 0$. So $X \times Y$ is the affine variety $\operatorname{Spec} T$, where $T = \mathbb{C}[u, v]/((f(u), g(v))$. The projections from $X \times Y$ to X and Y correspond to the algebra homomorphisms $A \to T$ and $B \to T$ that sends $u \rightsquigarrow u$ and $v \rightsquigarrow v$. So thos e projctions are morphisms. Moreover, morphisms from an affine variety $Z = \operatorname{Spec} R$ to $X \times Y$ correspond to homomorphisms $R \to T$, and as shown in Proposition 2.8.8, such homomorphisms correspond to solutions in R of the defining equations $f(u) = 0$ and $g(v) = 0$ of T. They also correspond bijectively to pairs of homomorphisms $R \to A$ and $R \to B$.

Next, Lemma 3.5.3 shows that it suffices to prove **(i)** and **(ii)** when X and Y are projective spaces, say $\mathbb{P}^m$ and $\mathbb{P}^n$. So we suppose that this is the case.

Let $\mathbb{U}^i$ and $\mathbb{V}^j$ be the standard affine open subsets of $\mathbb{P}^m$ and $\mathbb{P}^n$, respectively. The $\mathbb{U}^i \times \mathbb{V}^j$ is an affine open subset of the product. By what we know about products of affine varieties, the projections from $\mathbb{U}^i \times \mathbb{V}^j$ to $\mathbb{P}^m$ and $\mathbb{P}^n$ are morphisms. Lemma 3.5.3 **(iii)** shows that the projections from $\mathbb{P}^m \times \mathbb{P}^n$ to $\mathbb{P}^m$ and $\mathbb{P}^n$ are morphisms too.

Say that we have morphisms $Z \xrightarrow{f} \mathbb{P}^m$ and $Z \xrightarrow{g} \mathbb{P}^n$. We cover $\mathbb{P}^m$ and $\mathbb{P}^n$ by the standard affine open sets $\mathbb{U}^i$ and $\mathbb{V}^j$. Let Z^{ij} be the intersection of the inverse images of $\mathbb{U}^i$ and $\mathbb{V}^j$ in Z. These sets cover Z, and we may cover each Z^{ij} by affine open sets $Z_{ij\nu}$. The mapping property for products of affine varieties tells us that the map $Z_{ij\nu} \to \mathbb{U}^i \times \mathbb{V}^j$ is a morphism, and Lemma 3.5.3 **(iii)** shows that the map $Z \to \mathbb{P}^m \times \mathbb{P}^n$ is a morphism. $\square$