Chapter 3 PROJECTIVE GEOMETRY

pvariety 3.1 Projective Varieties

As before, the projective space $\mathbb{P}^n$ of dimension n is the set of equivalence classes of nonzero vectors $(x_0, ..., x_n)$, the equivalence relation being that for any nonzero complex number λ,

xlambdax (3.1.1)
$$(x_0, ..., x_n) \sim (\lambda x_0, ..., \lambda x_n).$$

The polynomial ring $\mathbb{C}[x_0, ..., x_n]$ will be denoted by $\mathcal{R}$ here. A polynomial $f(x_0, ..., x_n)$ *vanishes* at a point $x = (x_0, ..., x_n)$ of $\mathbb{P}^n$ if $f(\lambda x) = 0$ for every $\lambda \neq 0$, and this happens if and only if each of the homogeneous parts of f vanishes (**??**). So when studying loci of polynomial equations in projective space, we restrict attention to families of homogeneous polynomials.

homogideal **3.1.2. Proposition.** *The following conditions on an ideal of $\mathcal{R}$ are equivalent:*

- *$\mathcal{I}$ is generated by homogeneous polynomials.*
- *If a polynomial f is in $\mathcal{I}$, then its homogeneous parts are in $\mathcal{I}$.*

An ideal $\mathcal{I}$ of $\mathcal{R}$ is homogeneous if it satisfies these conditions. □

The maximal ideal $\mathcal{M} = (x_0, ..., x_n)$ of $\mathcal{R}$ generated by the variables is special because it has no zeros in projective space. It is sometimes called the *irrelevant ideal*.

nozeros **3.1.3. Proposition.** *Let $\mathcal{I}$ be the ideal of $\mathcal{R}$ generated by homogeneous polynomials $g_1, ..., g_k$. The locus of zeros $g_1 = \cdots = g_k = 0$ in projective space is empty if and only if, either the radical of $\mathcal{I}$ is the irrelevant ideal $\mathcal{M}$, or $\mathcal{I}$ is the unit ideal.*

For the proof, we will want to look at the affine space $\mathbb{A}^{n+1}$ with coordinates $x_0, ..., x_n$, and at the map from the complement of the origin in that affine space to the corresponding point of projective space $\mathbb{P}^n$

proof. Let Z be the locus of zeros of $\mathcal{I}$ in $\mathbb{P}^n$. If $\operatorname{rad}\mathcal{I} = \mathcal{M}$, then $\mathcal{I}$ contains a power of each variable. Powers of the variables have no common zeros in $\mathbb{P}^n$, so Z is empty. Suppose that Z is empty, and let W be the locus of zeros of $\mathcal{I}$ in the affine space $\mathbb{A}^{n+1}$ with coordinates $x_0, ..., x_n$. If $a = (a_0, ..., a_n)$ is a point of W other than the origin, and if $\mathcal{I}$ vanishes at a, then the point of $\mathbb{P}^n$ with that coordinate vector is in Z. Since there is no such point, the origin is the only possible point of W. If W is the one point space consisting of the origin, the Strong Nullstellensatz tells us that the radical of $\mathcal{I}$ is $\mathcal{M}$, and if W is empty, $\mathcal{I}$ will be the unit ideal. □

fadicalhomoogeneous **3.1.4. Lemma.** **(i)** *The radical of a homogeneous ideal is homogeneous.*

(ii) *A homogeneous ideal $\mathcal{P}$ that isn't the unit ideal is a prime ideal if and only if, whenever f and g are homogeneous polynomials whose product fg is in $\mathcal{P}$, either f is in $\mathcal{P}$ or g is in $\mathcal{P}$. In other words, a homogeneous ideal is a prime ideal if the condition for a prime ideal is satisfied when the polynomials are homogeneous.*

(iii) *If a homogeneous radical ideal $\mathcal{I}$ is not a prime ideal, there are homogeneous radical ideals $\mathcal{A}, \mathcal{B}$ that properly contain $\mathcal{I}$, $\mathcal{I} < \mathcal{A}$ and $\mathcal{I} < \mathcal{B}$, such that $\mathcal{I} = \mathcal{A} \cap \mathcal{B}$.*

proof. **(i)** Let $\mathcal{I}$ be a homogeneous ideal, and let f be an element of its radical. So f^r is in $\mathcal{I}$ for some r. We write $f = f_0 + \cdots + f_d$ as a sum of its homogeneous parts. The highest degree term of f^r is $(f_d)^r$. Since $\mathcal{I}$ is homogeneous, $(f_d)^r$ is in $\mathcal{I}$ and f_d is in $\operatorname{rad}\mathcal{I}$. Then $f_0 + \cdots + f_{d-1}$ is also in $\operatorname{rad}\mathcal{I}$. By induction on d, all of the homogeneous parts $f_0, ..., f_d$ are in $\operatorname{rad}\mathcal{I}$.

(ii) Suppose that a homogeneous ideal $\mathcal{P}$ satisfies the prime ideal condition for homogeneous polynomials. Let f and g be arbitrary polynomials whose product fg is in $\mathcal{P}$. If f has degree d and g has degree e, the highest degree part of the product fg is $f_d g_e$. It is in the homogeneous ideal $\mathcal{P}$. Therefore one of the factors, f_d or g_e, say f_d, is in $\mathcal{P}$. Let $f' = f - f_d$. Then $f'g$ is in $\mathcal{P}$, and it has lower degree than fg. By induction on the degree of fg, f' or g is in $\mathcal{P}$, and if f' is in $\mathcal{P}$, so is f.

(iii) If a homogeneous ideal $\mathcal{I}$ is not a prime ideal, there are homogeneous polynomials f and g not in $\mathcal{I}$ whose product fg is in $\mathcal{I}$. Let $\mathcal{A} = \mathcal{I} + (f)$ and $\mathcal{B} = \mathcal{I} + (g)$. Then $\mathcal{I} \subset \mathcal{A} \cap \mathcal{B}$. The product ideal $\mathcal{AB}$ is contained in $\mathcal{I}$. Since $(\mathcal{A} \cap \mathcal{B})^2 \subset \mathcal{AB}$ and since $\mathcal{I}$ is a radical ideal, $\mathcal{A} \cap \mathcal{B} \subset \mathcal{I}$. □

A subset of projective space $\mathbb{P}^n$ is *closed* in the Zariski topology if it is the set of zeros of some homogeneous polynomials in $\mathcal{R}$, or the set of zeros of a homogeneous ideal of $\mathcal{R}$. The fact that $\mathcal{R}$ is noetherian implies that $\mathbb{P}^n$ is a noetherian space. Every closed set is a finite union of irreducible closed sets (**??**).

We denote the set of zeros in $\mathbb{P}^n$ of a homogeneous ideal $\mathcal{I}$ by $V(\mathcal{I})$, and the set of zeros of a homogeneous polynomial f by $V(f)$.

homstrnull **3.1.5. Corollary.** *(projective Strong Nullstellensatz) Let g be a homogeneous polynomial in $x_0, ..., x_n$ that isn't a constant, and let $\mathcal{I}$ be a homogeneous ideal in $\mathbb{C}[x]$. If g vanishes at every point of $V(\mathcal{I})$, then $\mathcal{I}$ contains a power of g.*

proof. Let X be the locus of zeros of $\mathcal{I}$ in the affine space $\mathbb{A}^{n+1}$ with coordinates x. The polynomial g vanishes at every point of X different from the origin. And since g is not a constant, it vanishes at the origin too. So the affine Strong Nullstellensatz **??** applies. □

gdividesf **3.1.6. Corollary.** *Let g be a homogeneous polynomial and let f be an irreducible homogeneous polynomial. If $V(f) \subset V(g)$, then f divides g.* □

The set $V(\mathcal{P})$ of zeros of a homogeneous prime ideal $\mathcal{P}$, not the irrelevant ideal, is a *projective variety*.

closedirred **3.1.7. Lemma.** *The projective varieties are the irreducible closed subsets of projective space.*

The proof is analogoues to the proof of Lemma **??**. □

zartopprod (**3.1.8**) The Zariski topology on a product

Closed subsets of a product of projective spaces were defined in **??**. A subset of $\mathbb{P}^m \times \mathbb{P}^n$ is *closed* if it is the set of zeros of a system of *bihomogeneous* polynomial equations $f(x, y) = 0$, polynomial equations that are homogeneous in x and also in y. The Zariski topology on a product $X \times Y$ of projective varieties is the topology induced from the product of projective spaces. Several examples of closed subsets of product varieties were given in Chapter **??**, (**??**).

It is important to note that the Zariski topology on a product is much finer than the product topology. For example, the proper closed subsets of $\mathbb{P}^1$ are the nonempty finite subsets, so in the product topology, the closed subsets of $\mathbb{P}^1 \times \mathbb{P}^1$ are finite unions of points and sets of the form $\mathbb{P}^1 \times p$ and $p \times \mathbb{P}^1$ ('horizontal' and 'vertical' lines). Most loci $\{f = 0\}$, with f bihomogeneous, aren't of this form. The *diagonal* in $\mathbb{P}^1 \times \mathbb{P}^1$, the set of points (p, p), which is defined by the bihomogeneous equation $x_0y_1 - x_1y_0 = 0$, is one example.

homogen

3.2 Homogenizing and Dehomogenizing

As before, the standard affine subset U^i of $\mathbb{P}^n$ is the set of points at which the coordinate x_i is nonzero. The subsets U^i are n-dimensional affine spaces that form the standard affine cover of $\mathbb{P}^n$. In this section, we discuss the relation between closed subsets of $\mathbb{P}^n$ and closed subsets of the standard affine spaces. We use the standard affine U^0 as an illustration, remembering that the index 0 can be replaced by any other index. We recall that $U^0 = \operatorname{Spec} \mathbb{C}[u_1, ..., u_n]$, where $u_i = x_i/x_0$.

Let $f(x_0, ..., x_n)$ be a homogeneous polynomial. Its *dehomogenization*, with respect to the index 0, is the non-homogeneous polynomial obtained by setting $x_0 = 1$:

dehomog (3.2.1)
$$F(x) = f(1, x_1, ..., x_n).$$

For example, the dehomogenization of $x_0^3 + x_0x_1^2 + x_1x_2^2$ is $1 + x_1^2 + x_1x_2^2$.

The substitution $x_0 = 1$ defines a surjective homomorphism $\mathbb{C}[x_0, ..., x_n] \xrightarrow{\epsilon} \mathbb{C}[x_1, ..., x_n]$, so the process of dehomogenizing is compatible with the algebra structure.

Let $F(x_1, ..., x_n)$ be a non-homogeneous polynomial. Its *homogenization* is the homogeneous polynomial $f(x_0, ..., x_n)$ obtained as follows: Substitute $x_k = u_k$, where $u_k = x_k/x_0$, then multiply by the smallest power of x_0 required to clear the denominator. Or, multiply each monomial that appears in F by a suitable power of x_0 to bring its degree up to the degree d of F. For example, the homogenization of $1 + x_1^2 + x_1x_2^2$ is $x_0^3 + x_0x_1^2 + x_1x_2^2$.

dehomequal **3.2.2. Lemma.** **(i)** *The homogenization of a polynomial F is the unique homogeneous polynomial that isn't divisible by x_0, whose de-homogenization is F.*

(ii) *Homogeneous polynomials f and g whose dehomogenizations are equal differ by a power of x_0.*

(iii) *If two homogeneous polynomials f and g have no common (nonconstant) factor, neither do their dehomogenizations F and G.* □

Thus homogenization and dehomogenization are almost inverse operations:

$$\textit{dehomog}(\textit{homog}(F)) = F \quad \text{and if } x_0 \text{ doesn't divide } f\text{, then} \quad \textit{homog}(\textit{dehomog}(f)) = f$$

The *dehomogenization* I_0 of a homogeneous ideal $\mathcal{I}$ in $\mathbb{C}[x_0, ..., x_n]$, with respect to the index 0, is the set of dehomogenizations of elements of $\mathcal{I}$, the image of $\mathcal{I}$ via the above homomorphism ϵ. If the dehomogenization G of a homogeneous polynomial g is in I_0, then $x_0^k g$ will be in $\mathcal{I}$ for sufficiently large k.

If a point p of $\mathbb{P}^n$ has coordinates $(1, a_1, ..., a_n)$, its coordinates in the standard affine space U^0 will be $(a_1, ..., a_n)$. So if G is the dehomogenization of a homogeneous polynomial g, then g vanishes at p if and only if G vanishes at the same point, viewed as a point of U^0. Consequently, if $\mathcal{Z}$ is the set of zeros of a homogeneous ideal $\mathcal{I}$ in $\mathbb{P}^n$, the set of zeros in U^0 of the dehomogenized ideal I_0 will be the intersection $\mathcal{Z} \cap U^0$.

The *homogenization* of an ideal I of $\mathbb{C}[x_1, ..., x_n]$ is the ideal $\mathcal{I}$ *generated by* the homogenizations of the elements of I. It consists of the products $x_0^k f$, where f is the homogenization of an element F of I, and $k \geq 0$.

cIisclosure **3.2.3. Proposition.** *If Z is the set of zeros of an ideal I in the affine space U^0, the set of zeros of its homogenization $\mathcal{I}$ is the closure of Z in $\mathbb{P}^n$.*

proof. Let Y be the zero set of the homogenized ideal $\mathcal{I}$. As we have seen, $Y \cap U^0 = Z$. Since Y is closed, it contains the closure $\overline{Z}$ of Z in $\mathbb{P}^n$ (see Section **??**). By definition, a homogeneous polynomial will vanish on $\overline{Z}$ if and only if it vanishes on Z. We must show that if a homogeneous polynomial f vanishes on Z, then it vanishes on Y. We write $f = x_0^k g$, where x_0 doesn't divide g. Then g vanishes on Z too, and it suffices to show that g vanishes on Y. The dehomogenization G of g also vanishes on Z. So for large n, G^n is in I. The homogenization of G^n is g^n. So g^n is in $\mathcal{I}$. Then g^n and g vanish on Y. □

standard-isopen **3.2.4. Corollary.** *The Zariski topology on the standard affine open set U^0 is the topology induced from the Zariski topology on $\mathbb{P}^n$.* □

homogprime **3.2.5. Proposition.** *Let $\mathcal{P}$ be a homogeneous prime ideal in $\mathcal{R} = \mathbb{C}[x_0, ..., x_n]$. Its dehomogenization P with respect to the index 0 is either a prime ideal or the unit ideal. Therefore if X is a variety in $\mathbb{P}^n$, then either $X \cap U^0$ is an affine variety, or it is empty.*

proof. Suppose that the dehomogenization P of $\mathcal{P}$ is not the unit ideal. Since the dehomogenization of x_0 is 1, x_0 is not in $\mathcal{P}$. Let F and G be polynomials whose product FG is in P, and let f, g be homogeneous polynomials whose dehomogenizations are F, G, respectively. The dehomogenization of fg is FG, which is in P. Therefore $x_0^k fg$ is in $\mathcal{P}$ for some k. Since $\mathcal{P}$ is a prime ideal, one of the factors x_0, f, or g is in $\mathcal{P}$. Since x_0 is not in $\mathcal{P}$, f or g is in $\mathcal{P}$, and then F or G is in P. □

closedinaff **3.2.6. Proposition.** **(i)** *If $\{U^i\}$ is a covering of a topological space X by open subsets, a subset X of S is closed if and only if $X \cap U^i$ is closed in U^i for every i.*
(ii) *Let $\{U^i\}$ be the standard affine cover of $\mathbb{P}^n$. A subset X of $\mathbb{P}^n$ is closed if and only if $X \cap U^i$ is closed in U^i for every i.*

This is just topology. □

By the way, when we say that the sets $\{U^i\}$ cover a topological space X, we mean that $X = \bigcup U^i$. We don't allow U^i to contain elements that aren't in X, though that would be customary usage in the english language.

someprojvar

3.3 Some Projective Varieties.

quadric-surface **(3.3.1) a quadric surface**

The product $\mathbb{P}^1 \times \mathbb{P}^1$ of projective lines maps bijectively to a quadric in $\mathbb{P}^3_w$. Let coordinates in the two copies of $\mathbb{P}^1$ be (x_0, x_1) and (y_0, y_1), respectively, and let the coordinates in $\mathbb{P}^3$ be w_{ij}, with $0 \le i, j \le 1$. The map is defined by $w_{ij} = x_i y_j$, and its image is the quadric Q:

ponepone **(3.3.2)**
$$w_{00}w_{11} = w_{01}w_{10}$$

The fact that f is bijective is left as an exercise.

verone-seemb **(3.3.3) the Veronese embedding** (Giuseppe Veronese 1854-1917)

Let the coordinates in $\mathbb{P}^n$ be y_i, and let those in $\mathbb{P}^N$ be w_{ij}, $0 \le i, j \le n$. So $N = (n+1)^2 - 1$. The *Veronese embedding* is the map $\mathbb{P}^n \xrightarrow{u} \mathbb{P}^N$ defined by $w_{ij} = y_i y_j$. It is injective, and its image in $\mathbb{P}^N$ is the locus V of the equations

veroneq (3.3.4)
$$w_{ij}w_{k\ell} = w_{i\ell}w_{kj} \quad \text{and} \quad w_{ij} = w_{ji}, \quad \text{for } 0 \le i, j, k, \ell \le n$$

This is shown as follows: Let W^{ij} denote the standard affine subset of $\mathbb{P}^N$ of points at which $w_{ij} \neq 0$, and let $V^{ij} = V \cap W^{ij}$. The equation $w_{ii}w_{jj} = w_{ij}^2$ shows that $V^{ij} = V^{ii} \cap V^{jj}$. Therefore the open sets V^{jj} cover V. The inverse image of V^{jj} is the standard affine open subset U^j: $\{y_j \neq 0\}$ of $\mathbb{P}^n$.

The equations defining V hold when $w_{ij} = y_i y_j$, so u sends $\mathbb{P}^n$ to V. If $y_0 = 1$, for instance, then $y_j = w_{0j}$. So u is injective. Suppose given a point w of V. Some coordinate w_{ii}, say w_{00}, must be nonzero. Then the equations hold with $y_0 = 1$, $y_i = w_{0i}$, and $w_{00} = 1$. So w is in the image of u.

When redundancy is removed by setting $w_{10} = w_{01}$, the Veronese embedding sends the projective line $\mathbb{P}^1$ to the conic in $\mathbb{P}^2$ whose equation is $w_{00}w_{11} = w_{01}^2$.

There are higher order Veronese embeddings, defined in the analogous way by the monomials of some degree $d > 2$.

We will define the concept of an *isomorphism* of varieties in Section 3.7. The maps $\mathbb{P}^1 \times \mathbb{P}^1 \to Q$ and $\mathbb{P}^n \to V$ described above are isomorphisms.

defprojection **(3.3.5) projection**

Let π be the operation that drops the last coordinate of a point: $\pi(x_0, ..., x_n) = (x_0, ..., x_{n-1})$. This operation is called a projection

projection (3.3.6)
$$\pi : \mathbb{P}^n \to \mathbb{P}^{n-1}$$

It is defined at all points of $\mathbb{P}^n$ except at the *center of projection* $q = (0, ..., 0, 1)$.

To analyze the geometry of the projection, we denote the coordinates in $\mathbb{P}^n$ and $\mathbb{P}^{n-1}$ by $x = x_0, ..., x_n$ and $y = y_0, ..., y_{n-1}$, respectively. The fibre $\pi^{-1}(y)$ over a point y is the set of points (y, x_n) with x_n arbitrary. It is the line ℓ_y through the points $(y, 0)$ and $q = (0, 1)$, with the point q omitted.

Let Γ be the locus in $\mathbb{P}^n \times \mathbb{P}^{n-1}$ defined by the bihomogeneous equations $x_i y_j = x_j y_i$, $0 \le i, j \le n-1$. When we project $\mathbb{P}^n \times \mathbb{P}^{n-1}$ and its subset Γ to $\mathbb{P}^{n-1}$, the fibre Γ_y of Γ over a point y of $\mathbb{P}^{n-1}$ is the set $\ell_y \times y$ of points (x, y) with x in ℓ_y. When we project in the other direction, to $\mathbb{P}^n$, the fibre over a point $(x_0, ..., x_n)$ distinct from q is a single point $(x, \pi(x))$. The fibre over q is the set $q \times \mathbb{P}^{n-1}$, a projective space.

Because the point q of $\mathbb{P}^n$ is replaced in Γ by a projective space, the map $\Gamma \to \mathbb{P}^n$ is called a *blowing up* of the point q of $\mathbb{P}^n$.

figure

diagonal **(3.3.7) the diagonal**

If X is a set, the *diagonal* X_Δ in the product $X \times X$ is the set of points (x, x).

diagclosed **3.3.8. Proposition.** *Let X be an affine or a projective variety. The diagonal X_Δ is a closed subvariety of the product $X \times X$.*

proof. We do the case of a projective variety. Let $x'_0, ..., x'_n$ and $x''_0, ..., x''_n$ be coordinates in the two factors of the product $\mathbb{P} \times \mathbb{P}$ $(\mathbb{P} = \mathbb{P}^n)$. The diagonal $\mathbb{P}_\Delta$ in $\mathbb{P} \times \mathbb{P}$ is the closed subvariety defined by the equations $x'_i x''_j = x'_j x''_i$. Next, suppose that X is the closed subvariety of $\mathbb{P}$ defined by a system of homogeneous equations $f(x) = 0$. The diagonal X_Δ can be identified as the intersection of $X \times X$ with the diagonal $\mathbb{P}_\Delta$, so it is a closed subvariety of $X \times X$. The system of bihomogeneous equations $f(x') = 0$, $f(x'') = 0$ defines the product $X \times X$ as a closed subset of $\mathbb{P}^n \times \mathbb{P}^n$, so X_Δ can also be described as the closed subvariety of $\mathbb{P} \times \mathbb{P}$ defined by the system of equations

Xdelta (3.3.9)
$$x'_i x''_j = x'_j x''_i, \;\; f(x') = 0, \;\; f(x'') = 0.$$

□

It is interesting to compare Proposition 3.3.8 with the Hausdorff condition for a topological space. Recall that a topological space X is a *Hausdorff space* if distinct points have disjoint open neighborhoods. A variety with its Zariski topology isn't a Hausdorff space unless it consists of a single point.

The *product topology* on a product $X \times Y$ of topological spaces is the topology whose closed sets are products $C \times D$ of closed sets C and D in X and Y, respectively.

The proof of the next lemma is an exercise in topology:

hausdorff-diagonal **3.3.10. Lemma.** *A topological space X is a Hausdorff space if and only if, when $X \times X$ is given the product topology, the diagonal map $X \xrightarrow{\Delta} X \times X$ defined by $\Delta(x) = (x, x)$ embeds X as a closed subset X_Δ of $X \times X$.* □

There is no contradiction with Proposition 3.3.8, because the Zariski topology on the product $X \times X$ is finer than the product topology (see **3.1.8**).

segreemb (**3.3.11**) **the Segre embedding** (Corrado Segre 1863-1924)

The product $\mathbb{P}^m_x \times \mathbb{P}^n_y$ of projective spaces is embedded by the *Segre map* into another projective space $\mathbb{P}^N_w$ with coordinates w_{ij}, $i = 0, ..., m$ and $j = 0, ..., n$. So $N = (m+1)(n+1)-1$. The Segre map generalizes the map of $\mathbb{P}^1 \times \mathbb{P}^1$ to a quadric that was described in (**3.3.2**). It is defined by

segrecoords (3.3.12)
$$w_{ij} = x_i y_j.$$

segreequations **3.3.13. Proposition.** *The Segre map is injective, and its image in $\mathbb{P}^N$ is the locus of solutions of the equations*

segreequations (3.3.14)
$$w_{ij}w_{k\ell} - w_{i\ell}w_{kj} = 0.$$

proof. Let's call Π the locus of these equations. When one substitutes (3.3.12) into the equations (3.3.14), they become $x_iy_jx_ky_\ell - x_iy_\ell x_ky_j = 0$, and they are true at every point of $\mathbb{P}^m \times \mathbb{P}^n$. So the image of the Segre map is contained in Π.

The standard affine open subsets of $\mathbb{P}^N$ are the sets $W^{ij} : \{w_{ij} \neq 0\}$. Say that we have a point p of Π at which $w_{00} \neq 0$. We set $w_{00} = 1$. Then $w_{ij} = w_{i0}w_{0j}$, which shows that the coordinates w_{i0} cannot all be zero, and that the same is true of the coordinates w_{0j}. Then the point (x, y) with $x_0 = y_0 = 1$, $x_i = w_{i0}$, and $y_j = w_{0j}$ is the unique point of $\mathbb{P}^m \times \mathbb{P}^n$ whose image is p. This shows that $U^0 \times U'^0$ maps bijectively to $\Pi^{00} = \Pi \cap W^{00}$. □

minorsprime **3.3.15. Note.** The left sides of the equations (3.3.14) are the 2×2 minors (the determinants of the 2×2 submatrices) of the matrix (w_{ij}). A matrix M has rank 1 if and only if all of its 2×2 minors vanish. Thus Π can be identified as the set of rank 1 matrices. It is a fact that for any $r > 0$, the $r \times r$ minors of an $m \times n$-matrix with variable entries generate a prime ideal. This fact isn't easy to prove, and we won't use it. However, the next lemma shows that the radical of the ideal $\mathcal{P}$ of $\mathbb{C}[w]$ generated by the left sides of (3.3.14) is a prime ideal. So if a homogeneous polynomial $f(w)$ vanishes on Π, then some power of f is in $\mathcal{P}$. □

closedinpxp **3.3.16. Proposition.** *A subset of $\mathbb{P}^m \times \mathbb{P}^n$ is closed if and only if its image via the Segre map is closed in $\mathbb{P}^N$. Therefore the map from $\mathbb{P}^m \times \mathbb{P}^n$ to its Segre image is a homeomorphism.*

proof. Let Y be a closed subset of Π, the set of zeros of some homogeneous polynomials $f(w)$ in the variables w_{ij}. Substituting $w_{ij} = x_iy_j$ into any one of the equations $f = 0$, we obtain a bihomogeneous polynomial $\widetilde{f}(x, y) = f(x_iy_j)$, of the same degree as f in x and in y. Thus the zeros of f are images of the zeros of $\widetilde{f}$. So the inverse image of Y in $\mathbb{P}^m \times \mathbb{P}^n$ is closed.

Conversely, let X be a subset of $\mathbb{P}^m \times \mathbb{P}^n$, defined by a bihomogeneous polynomial $g(x, y)$. Say that g has degree r in x and degree s in y. If $r = s$, we may collect the variables in pairs x_iy_j and replace each such pair by w_{ij}, to obtain a homogeneous polynomial in w whose zeros, together with the defining equations (3.3.14), give us the Segre image of X. Suppose that $r \geq s$. Let $k = r - s$. Because the variables y cannot all be zero at any point of $\mathbb{P}^n$, X is also the set of zeros of the system of equations $y_0^kg = y_1^kg = \cdots = y_n^kg = 0$, and these polynomials are bihomogeneous, of the same degree in x and y. □

linesinp 3.4 Lines in Projective Three-Space

The *Grassmanian* $G(m, n)$ is a variety that parametrizes subspaces of dimension m of the vector space $\mathbb{C}^n$, or linear spaces of dimension $m-1$ in $\mathbb{P}^{n-1}$. For example, the Grassmanian $G(1, n+1)$ is the projective space $\mathbb{P}^n$. Points of $\mathbb{P}^n$ correspond to one-dimensional subspaces of $\mathbb{C}^{n+1}$. In this section we describe the Grassmanian $G(2, 4)$, which parametrizes lines in $\mathbb{P}^3$ or two-dimensional subspaces of $V = \mathbb{C}^4$. We denote $G(2, 4)$ by $\mathbb{G}$. The point of $\mathbb{G}$ that corresponds to a line ℓ in $\mathbb{P}^3$ will be denoted by $[\ell]$.

One can use row reduction to get insight into the structure of $\mathbb{G}$. A two-dimensional subspace U of $V = \mathbb{C}^4$ will have a basis (u_1, u_2). Let M be the 2×4 matrix whose rows are u_1, u_2. The rows of the matrix obtained from M by row reduction span the same space U, and the row-reduced matrix is uniquely determined by the subspace. Provided that the left hand 2×2 submatrix of M is invertible, the row-reduced matrix will have the form

rowreduced (3.4.1)
$$M' = \begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{pmatrix}$$

So the Grassmanian $\mathbb{G}$ contains as open subset a four-dimensional affine space whose coordinates are the four variable entries of this matrix.

In any 2×4 matrix M with independent rows, some pair of columns will be independent, and those columns can be used in place of the first two in a row reduction. So $\mathbb{G}$ is covered by six four-dimensional affine spaces that we denote by W^{ij}, $1 \le i < j \le 4$, W^{ij} being the space of 2×4 matrices such that $column_i = e_1$ and $column_j = e_2$. Thus the Grassmanian seems somewhat similar to $\mathbb{P}^4$. It isn't the same space. One difference is that $\mathbb{P}^4$ is covered by five four-dimensional affine spaces.

extalgone **(3.4.2) the exterior algebra**

Let V be a complex vector space. The *exterior algebra* $\bigwedge V$ (read 'wedge V') is a ring that contains the complex numbers and is generated by the elements of V, with either of the two equivalent relations

extalg (3.4.3)
$$vv = 0, \quad \text{or} \quad vw = -wv,$$

for all v, w in V.

Suppose that V has dimension four. Let $(v_1, ..., v_4)$ be a basis of V, and let $\bigwedge^i V$ denote the subspace of $\bigwedge V$ spanned by products of length i of elements of V. Then

extbasitwos (3.4.4)

$\bigwedge^0 V = \mathbb{C}$ is a space of dimension 1, with basis $\{1\}$,
$\bigwedge^1 V = V$ is a space of dimension 4, with basis $\{v_1, v_2, v_3, v_4\}$,
$\bigwedge^2 V$ is a space of dimension 6, with basis $\{v_iv_j \,|\, i < j\} = \{v_1v_2, v_1v_3, v_1v_4, v_2v_3, v_2v_4, v_3v_4\}$,
$\bigwedge^3 V$ is a space of dimension 4, with basis $\{v_iv_jv_k \,|\, i < j < k\} = \{v_1v_2v_3, v_1v_2v_4, v_1v_3v_4, v_2v_3v_4\}$,
$\bigwedge^4 V$ is a space of dimension 1, with basis $\{v_1v_2v_3v_4\}$,
$\bigwedge^q V = 0$ if $q > 4$.

Because it is the direct sum of the subspaces $\bigwedge^i V$ and because multiplication maps $\bigwedge^i V \times \bigwedge^j V$ to $\bigwedge^{i+j} V$, the exterior algebra $\bigwedge V$ is an example of a *noncommutative graded algebra.*

To familiarize yourself with computation in $\bigwedge V$, verify that $v_3v_2v_1 = -v_1v_2v_3$ and that $v_3v_1v_2 = v_1v_2v_3$.

We write an element of $\bigwedge^2 V$ as

wedgetwo (3.4.5)
$$w = \sum_{i<j} a_{ij} v_i v_j.$$

Speaking informally, we regard $\bigwedge^2 V$ as an affine space of dimenion 6, identifying w with the vector whose coordinates are the six coefficients a_{ij}. We also use the symbol w to denote the point of the projective space $\mathbb{P}^5$ with the same coordinates.

An element w of $\bigwedge^2 V$ is *decomposable* if it is a product of two elements of V.

describede-comp **3.4.6. Proposition.** **(i)** *The decomposable elements w of $\bigwedge^2 V$ are those such that $ww = 0$, and the relation $ww = 0$ is given by the following equation in the coefficients a_{ij}:*

eqgrass (3.4.7)
$$a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23} = 0.$$

(ii) *Suppose given a nonzero decomposable element w of $\bigwedge^2 V$, say $w = u_1u_2$, with u_i in V. The pair (u_1, u_2) is a basis for a two-dimensional subspace U of V.*

(iii) *Let (u_1, u_2) be a basis for a two-dimensional subspace U of V, and let $w = u_1u_2$. Sending $U \rightsquigarrow w$ defines a bijective map from set of two-dimensional subspaces to the locus in $\mathbb{P}^5$ defined by (3.4.7).*

Thus $\mathbb{G} = G(2,4)$ can be represented as a quadric in $\mathbb{P}^5$.

proof. **(i)** If $w = u_1u_2$, then $ww = -u_1^2u_2^2$, which is zero because $u_1^2 = 0$. For the converse, we compute ww when $w = \sum_{i<j} a_{ij}v_iv_j$. The answer is

$$ww = 2\big(a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}\big)v_1v_2v_3v_4.$$

To show that w is decomposable if $ww = 0$, it seems simplest to factor w explictly. Since the assertion is trivial when $w = 0$, we may suppose that some coefficient of w, say a_{12}, is nonzero. Then if $ww = 0$,

factorw (3.4.8)
$$w \;=\; \frac{1}{a_{12}}\big(a_{12}v_2 + a_{13}v_3 + a_{14}v_4\big)\big(-a_{12}v_1 + a_{23}v_3 + a_{24}v_4\big).$$

(ii) If an element w of $\bigwedge^2 V$ is decomposable, say $w = u_1u_2$, and if w is nonzero, then u_1 and u_2 must be independent. They span a two-dimensional subspace. Conversely, let (u_1, u_2) be a basis for a subspace U of dimension 2, and let $w = u_1u_2$. Changing the basis by a 2×2 invertible matrix P multiplies the product w by the determinant of P. So the point w of $\mathbb{P}^5$ that corresponds to U is determined uniquely.

Finally, let (u_1, u_2) and (u_1', u_2') be bases for distinct two-dimensional subspaces U and U'. At least three of the vectors $\{u_1, u_2, u_1', u_2'\}$ will be independent, and therefore $u_1u_2 \neq \lambda u_1'u_2'$ (see 3.4.4). □

We will use the *algebraic dimension* of a variety here, though this concept won't be studied until Chapter **??**. We refer to the algebraic dimension simply as the *dimension*. By definition, the dimension of a variety X is defined to be the length d of the longest chain $C_0 > C_1 > \cdots > C_d$ of closed subvarieties of X. (In a longest chain, C_0 will be the whole space X, and C_d will be a point.) The topological dimension of X, its dimension in the classical topology, is always twice the algebraic dimension. The Grassmanian $\mathbb{G}$, for example, has dimension 4.

pinlclosed **3.4.9. Proposition.** *Let $\mathbb{P}^3$ be the projective space associated to the vector space V. In the product $\mathbb{P}^3\times\mathbb{G}$, the locus $\Gamma = \{p, [\ell] \mid p \in \ell\}$ of pairs such that p lies on ℓ is a closed subset of dimension 5.*

proof. Say that ℓ is the line in $\mathbb{P}^3$ that corresponds to the subspace U with basis (u_1, u_2), and that p is represented by the vector x of V. Let $w = u_1u_2$. Then $p \in \ell$ means $x \in U$, which is true if and only if (x, u_1, u_2) is a dependent set, or if $xw = 0$. So Γ is the closed subset of $\mathbb{P}^3\times\mathbb{P}^5$ defined by the bihomogeneous equations $ww = 0$ and $xw = 0$. The fibre of Γ over the point $[\ell]$ of $\mathbb{G}$ is the set of points of ℓ. Thus Γ can be viewed as a four-dimensional family of lines, parametrized by $\mathbb{G}$. Its dimension is $4 + 1 = 5$. □

linesina-surface **(3.4.10) lines on a surface**

Suppose given a surface S in $\mathbb{P}^3$. We ask: Does S contain a line? One surface that contains lines is the quadric Q with equation $w_1w_2 = w_0w_3$, the image of the Segre map $\mathbb{P}^1\times\mathbb{P}^1 \to \mathbb{P}^3_w$ (**3.3.1**). It contains two families of lines, corresponding to the two "rulings" $p\times\mathbb{P}^1$ and $\mathbb{P}^1\times q$ of $\mathbb{P}^1\times\mathbb{P}^1$. There exist surfaces of arbitrary degree that contain lines, but we will see that a generic surface of degree four or more contains no line.

We use coordinates x_i with $i = 1,2,3,4$ for $\mathbb{P}^3$ here. There are $\binom{d+3}{3}$ monomials of degree d in four variables, so surfaces of degree d in $\mathbb{P}^3$ are parametrized by a projective space of dimension $N = \binom{d+3}{3} - 1$. Let $\mathbb{S}$ denote that projective space. The coordinates of the point $[S]$ of $\mathbb{S}$ that corresponds to a surface S are the coefficients of the monomials in its defining polynomial f. Speaking infomally, we say that a point of $\mathbb{S}$ "is" a surface of degree d in $\mathbb{P}^3$. (If f is reducible, its zero locus isn't a variety. Let's not worry about this.)

Consider the line ℓ_0 through the points $e_1 = (1,0,0,0)$ and $e_2 = (0,1,0,0)$. A surface $S : \{f = 0\}$ will contain ℓ_0 if and only if $f(x_1, x_2, 0, 0) = 0$ for all x_1, x_2. Substituting $x_3 = x_4 = 0$ into f leaves us with a polynomial in two variables:

scontainslzero (3.4.11)
$$f(x_1, x_2, 0, 0) = c_0x_1^d + c_1x_1^{d-1}x_2 + \cdots + c_dx_2^d,$$

where c_i are some of the coefficients of f. In order for this polynomial to be identically zero, all of its coefficients must be zero. So the surfaces that contain ℓ_0 correspond to the points of the linear subspace $\mathbb{L}_0$ of $\mathbb{S}$ defined by $c_0 = \cdots = c_d = 0$. This is a satisfactory answer to the question of which surfaces contain ℓ_0, and we can use it to make a guess about lines in a generic surface of degree d.

scontainslclosed **3.4.12. Lemma.** *In $\mathbb{G}\times\mathbb{S}$, the set Ξ of pairs $[\ell],[S]$ such that $\ell\subset S$ is a closed subset.*

proof. Let W^{ij}, $1\le i<j\le 4$ denote the six standard affine spaces that cover the Grassmanian. It suffices to show that the intersection $\Xi^{ij}=\Xi\cap(W^{ij}\times\mathbb{S})$ is closed in $W^{ij}\times\mathbb{S}$ (3.2.6). We inspect the case $i,j=1,2$.

A line ℓ such that $[\ell]$ is in W^{12} has a basis of the form u_1,u_2, where $u_1=(1,0,a_2,a_3)$ and $u_2=(0,1,b_2,b_3)$ for some uniquely determined a_i,b_i, and ℓ is the line $\{ru_1+su_2\}$. Let $f(x_1,x_2,x_3,x_4)$ be the defining polynomial of a surface S. Then ℓ is contained in S if and only if $\widetilde{f}(r,s)=f(r,s,ra_2+sb_2,ra_3+sb_3)$ is zero for all r and s, and $\widetilde{f}$ is a homogeneous polynomial of degree d in r,s. If we write $\widetilde{f}=z_0r^d+z_1r^{d-1}s+\cdots+z_ds^d$, the coefficients z_j will be polynomials in a,b and in the coefficients of f, and they will be homogeneous and linear in the coefficients of f. The zero locus of these coefficients is the closed set Ξ^{ij}. $\square$

The set of surfaces that contain our special line ℓ_0 is the linear subspace $\mathbb{L}_0$ of $\mathbb{S}$ defined by the vanishing of $d+1$ coordinates. So the dimension of $\mathbb{L}_0$ is $N-d-1$. The line ℓ_0 can be carried to any other line ℓ_1 by a linear map $\mathbb{P}^3\to\mathbb{P}^3$, so the sufaces that contain ℓ_1 also form a linear subspace of dimension $N-d-1$ in $\mathbb{S}$, and Ξ is the union of such linear spaces. The dimension of the Grassmanian $\mathbb{G}$ is 4. Therefore the dimension of Ξ is

dimspace-lines (3.4.13) $$\dim\Xi \;=\; (N-d-1)+4 \;=\; N-d+3 \;=\; \dim\mathbb{S}-d+3.$$

We project the product $\mathbb{G}\times\mathbb{S}$ and its subvariety Ξ to $\mathbb{S}$. The fibre of Ξ over a point $[S]$ is the set of pairs $[\ell],[S]$ such that ℓ is contained in S. It can be identified with the set of lines in S.

When the degree d is 1, $\dim\Xi=\dim\mathbb{S}+2$. Every fibre of Ξ over $\mathbb{S}$ will have dimension at least 2. In fact, every fibre has dimension equal to 2. Surfaces of degree 1 are planes, and the lines in a plane form a two-dimensional family.

When $d=2$, $\dim\Xi=\dim\mathbb{S}+1$. We can expect that most fibres of Ξ over $\mathbb{S}$ will have dimension 1. This is true: A smooth quadric contains two one-dimensional families of lines. (All smooth quadrics are equivalent with the quadric (**3.3.2**).) But if a quadratic polynomial $f(x_1,x_2,x_3,x_4)$ is the product of linear polynomials, its locus of zeros will be a union of planes. It will contain two-dimensional families of lines. Some fibres of Ξ over $\mathbb{S}$ have dimension 2.

When $d\ge 4$, $\dim\Xi<\dim\mathbb{S}$. The projection $\Xi\to\mathbb{S}$ cannot be surjective. Most surfaces of degree 4 or greater contain no lines.

The most interesting case is that $d=3$. Then $\dim\Xi=\dim\mathbb{S}$. Most fibres will have dimension zero. They will be finite sets. In fact, a generic cubic surface contains 27 lines. We have to wait to see why the number is precisely 27 (see (**??**)).

Our conclusions are intuitively plausible, but to be sure about them, we will need to study dimension carefully.

regfunction

3.5 Regular Functions.

As was explained in Chapter **??**, the regular functions on an affine variety $\operatorname{Spec} A$ are the functions defined by the elements of A. On an arbitrary variety X, a function is *regular* if its restrictions to the open sets of an affine covering (a covering by affine open subsets) are regular. But we have a problem: To say that an open subset is an affine variety means that it is isomorphic to a variety of the form $\operatorname{Spec} A$. This requires a definition of the term "isomorphism", and we don't yet have a definition. Moreover, for most varieties, the question of which open subsets are affine has no useful answer. This complicates matters a little.

prepremarks (**3.5.1**) **special affine open sets**

Let X be the affine variety $\operatorname{Spec} A$, and let $A_s=A[s^{-1}]$ be the simple localization obtained by inverting a nonzero element s of A. We have seen that $\operatorname{Spec} A_s$ is homeomorphic to an open subset of X, the complement X_s of the locus $\{s=0\}$ (**??**), and that this subset is called a *simple localization* of X. When regarding the algebra A_s as a localization of A, we identify $\operatorname{Spec} A_s$ with X_s and call it a *special affine open set*:

localXs (3.5.2) $$X_s \;=\; \{p\in X\mid s(p)\neq 0\}$$

Most affine varieties have many other open sets, but the next lemma shows that there are enough special affine opens.

enoughlocals **3.5.3. Lemma.** *On an affine variety X, the simple localizations X_s form a basis for the Zariski topology.*

proof. A set $\mathcal{B}$ of open subsets of a topological space X is a *basis* for the topology if every nonempty open set can be covered by open sets that are members of $\mathcal{B}$. We show that this is true when $\mathcal{B}$ is the set of simple localizations. Let V be any open set, and let p be a point of V. The complement of V is closed, so it is the set of common zeros of some elements of A. At least one of those elements, say s, will be nonzero at p. Then $p \in X_s \subset V$. □

We consider projective varieties next. Fortunately, we can pick out enough open subsets of a projective variety X to include among affine opens. Let U^i be the standard affine opens in the ambient projective space $\mathbb{P}^n$, as usual. We declare that U^i is the affine variety $\operatorname{Spec} R_i$, where R_i is the polynomial ring $\mathbb{C}[u]$, $u_j = x_j/x_i$. Then unless $X^i = X \cap U^i$ is empty, it will be a closed subvariety of U^i, and therefore it will be an affine variety (3.7.6). The sets X^i form the *standard affine covering* of X. Since the simple localizations of X^i form a basis for the topology on X^i, putting these localizations together for all i gives us a basis for the topology on X. Thus we have enough open sets that we know to be affine. We call them *special affine open sets*. Once we have a definition of an affine open set, the question of which ones are special won't be important. We can then forget about special affine open sets.

ratfnsdef **(3.5.4) rational functions**

The *function field* K of a (nonempty) affine variety $\operatorname{Spec} A$ is the field of fractions of its coordinate algebra A. The function field of a projective variety X is the function field of any one of its special affine open subsets. Because X is irreducible, the intersection of two nonempty open subsets is nonempty, and contains a special affine open set. This implies that the function fields of the special affine open sets are canonically isomorphic.

The elements of the function field K are called *rational functions* on X. A rational function needn't be a function on the whole space X, but it will define a function on some nonempty open subset.

defregfn **3.5.5. Definition.** A rational function α on a variety X is *regular* at a point p if there is a special affine open set $U = \operatorname{Spec} B$ that contains p, such that α is an element of B. A rational function is *regular* on a nonempty open subset V of X if it is regular at every point of V.

If p is a point of an affine variety $X = \operatorname{Spec} A$ then a rational function α on X that is regular at p can be written as a fraction a/s of elements of A such that $s(p) \neq 0$.

There is something to check, namely that when X is affine, this definition is equivalent to the one that was given in the previous chapter. The next proposition does this.

deloca **3.5.6. Proposition.** *Let X be the affine variety $\operatorname{Spec} A$. The regular functions on X, as defined above, are the elements of A.*

proof. An element of A is a regular function because X itself is a special affine open set. Conversely, let α be a regular function on X. Then for every point p of X, there is a simple localization $X_s = \operatorname{Spec} A_s$ such that α is an element of A_s. A finite set of these localizations, say $X_{s_1}, \ldots, X_{s_k}$, cover X. Then the elements s_i have no common zeros on X, and therefore they generate the unit ideal of A. The next lemma completes the proof.

sheafaxiomA **3.5.7. Lemma.** *Let K be the field of fractions of a domain A, and let $s_1, ..., s_k$ be nonzero elements of A that generate the unit ideal, An element α of K that is in A_{s_i} for every i is in A.*

proof. Since α is in A_{s_i}, we can write it as a fraction $\alpha = s_i^{-n} b_i$, with b_i in A. Then $s_i^n \alpha = b_i$. We can use the same exponent n for each i. Since the elements s_i generate the unit ideal of A, so do the powers s_i^n. Say that $\sum s_i^n r_i = 1$, with r_i in A. Then $\alpha = \sum s_i^n r_i \alpha = \sum r_i b_i$, and $r_i b_i$ is in A. □

ratfnspspace **(3.5.8) rational functions on projective space**

Let $\mathcal{R}$ denote the polynomial ring $\mathbb{C}[x_0, ..., x_n]$. Though it makes sense to say that a homogeneous polynomial f vanishes at a point of $\mathbb{P}^n$, a homogeneous polynomial doesn't define a function on $\mathbb{P}^n$. The reason is that, if f has degree d, then $f(\lambda x) = \lambda^d f(x)$. On the other hand, a fraction g/h of homogeneous polynomials of the same degree d defines a function wherever $h \neq 0$, because $g(\lambda x)/h(\lambda x) = (\lambda^d g(x))/(\lambda^d h(x)) = g(x)/h(x)$.

homogfractsfnfld **3.5.9. Lemma.** **(i)** *Rational functions on $\mathbb{P}^n$ can be represented as fractions g/h of homogeneous polynomials of the same degree.*

(ii) *Let g/h be a fraction of relatively prime homogeneous polynomials of degree d. The rational function α it defines is regular at a point x of $\mathbb{P}^n$ if and only if $h(x) \neq 0$.* $\square$

proof. **(ii)** Say that p is a point of the standard affine open U^0. We set $x_0 = 1$. Let the dehomogenizations of g and h be $\widetilde{g}(x) = g(1, x_1, ..., x_n)$ and $\widetilde{h}(x) = h(1, x_1, ..., x_n)$. The polynomials $\widetilde{g}, \widetilde{h}$ have no common factor, and $\alpha = \widetilde{g}/\widetilde{h}$. So α is regular at p if and only if $\widetilde{h}(x) \neq 0$. $\square$

openinpn **3.5.10. Corollary.** *Let h be a nonconstant, homogeneous polynomial of positive degree d, let Z be the set of zeros of h in $\mathbb{P}$, and let U be the complement of Z in $\mathbb{P}$. The nonzero rational functions that are regular on U are those of the form g/h^k, where h is a homogeneous polynomial of degree dk and $k \geq 0$.* $\square$

indeterminate **Note.** Let g and h be homogeneous polynomials, and let Y and Z be their zero sets in $\mathbb{P}$, respectively. The function $\alpha = g/h$ will tend to infinity as one approaches a point of Z that isn't also a point of Y. At intersections of Y and Z, α is indeterminate. $\square$

fnfld 3.6 The Structure Sheaf.

We have a good way to visualize a projective variety, as the zero locus of a homogeneous prime ideal in $\mathbb{P}^n$. We don't yet have a way to decide when two projective varieties are isomorphic, though some isomorphisms, such as those defined by the Veronese and the Segre embeddings, were exhibited empirically earlier in this chapter. It is rather obvious that an isomorphism should be a homeomorphism in the Zariski topology. This isn't sufficient. Since the proper closed subsets of a plane algebraic curve are the finite subsets, all plane curves are homeomorphic. The classical topology is better, because it controls the genus of a curve, but not all curves of a given genus should be called isomorphic.

Isomorphisms of projective varieties can rarely be defined directly by functions in the projective coordinates. The right definition of an isomorphism is a bijective map ψ such that both ψ and its inverse can be defined by regular functions.

lineconic **3.6.1. Example.** Let coordinates in $\mathbb{P}^1$ and $\mathbb{P}^2$ be x_0, x_1 and w_{00}, w_{01}, w_{11}, respectively. We define a map $\mathbb{P}^1 \xrightarrow{\psi} \mathbb{P}^2$ by $\psi(x_0, x_1) = (x_0^2, x_0x_1, x_1^2) = (w_{00}, w_{01}, w_{11})$. This map is obtained from the Veronese embedding (**3.3.3**) by eliminating the redundant coordinate w_{10}. Its image is the conic C whose equation is

$$w_{00}w_{11} = w_{01}^2.$$

The map ψ ought to be an isomorphism from $\mathbb{P}^1$ to C. But an isomorphism has an inverse, and because the coordinates of $\psi(x)$ are quadratic polynomials in x, the inverse isn't given globally by polynomials in w. One must piece together "local" maps.

The image of the standard open set $x_i \neq 0$ of $\mathbb{P}^1$ is the subset C^i: $w_{ii} \neq 0$, and the two opens sets C^0 and C^1 cover C. We define $C^0 \xrightarrow{\theta} \mathbb{P}^1$ $\theta(w) = (w_{00}, w_{01})$. Then

$$\psi\theta(w) = \psi(w_{00}, w_{01}) = (w_{00}^2, w_{00}w_{01}, w_{01}^2) = (w_{00}^2, w_{00}w_{01}, w_{00}w_{11}) \sim (w_{00}, w_{01}, w_{11}) = w$$

and

$$\theta\psi(x) = \theta(x_0^2, x_0x_1, x_1^2) = (x_0^2, x_0x_1) \sim (x_0, x_1) = x.$$

So θ inverts ψ on C^0. Similarly, $\eta(w) = (w_{01}, w_{11})$ inverts ψ on C^1. The maps θ and η and piece together to define the inverse of ψ. $\square$

The modern way to define the structure of a variety and to organize the process of piecing maps together uses structure sheaves. The structure sheaf of a variety X keeps track, implicitly, of the regular functions on all open subsets of X.

defstrsheaf **3.6.2. Definition.** The *structure sheaf* $\mathcal{O}_X$ on an affine or a projective variety X is the map defstrsheaftwo (3.6.3) $$\textit{(opens)}^\circ \xrightarrow{\mathcal{O}_X} \textit{(algebras)}$$

from open sets to algebras that sends a nonempty open set U to the algebra, denoted by $\mathcal{O}_X(U)$, of rational functions that are regular on U. So if U isn't empty, then $\mathcal{O}_X(U)$ is a subalgebra of the function field K of X. And by definition, $\mathcal{O}_X(\emptyset) = 0$.

Elements of $\mathcal{O}_X(U)$ are called *sections* of $\mathcal{O}_X$ over U, and elements of $\mathcal{O}_X(X)$ are *global sections*.

We make the set *(opens)* of open subsets of X into a category, defining morphisms between open sets to be inclusion maps. So if $V \subset U$ there is a unique morphism $V \to U$, and if $V \not\subset U$ there is no morphism $V \to U$. The structure sheaf $\mathcal{O}_X$ is a *sheaf* because it has the following two properties, both obvious:

strsheafprop (3.6.4)

- It is a contravariant *functor*: If $V \subset U$ are nonempty open sets, then $\mathcal{O}_X(V) \supset \mathcal{O}_X(U)$: A regular function on U is also regular on V. (The superscript $\circ$ in (3.6.3) indicates that arrows, inclusions here, are reversed by $\mathcal{O}_X$, i.e., that the functor is contraviariant.)

- It is a *sheaf*: If $V_1, ..., V_k$ are nonempty open subsets that cover another open set V, i.e., if $V = \bigcup V_i$, a function is regular on V if and only if it is regular on each V_i: $\mathcal{O}_X(V) = \bigcap \mathcal{O}_X(V_i)$.

This sheaf property will be discussed further in Chapter **??**.

pnminuspoint **3.6.5. Example.** Let Z be the locus of zeros of a homogeneous polynomial f of degree r. Corollary 3.5.9 describes the sections $\mathcal{O}_{\mathbb{P}^n}(U)$ of the structure sheaf of $\mathbb{P}^n$ on the complement U of Z. The sections on U are fractions h/f^k, where h is homogeneous, of degree rk. □

One never computes the regular functions on every open set U. We will see that, to determine the structure sheaf, it is enough to determine the regular functions on a collection of affine open sets that cover X, such as on the standard affine covering. Moreover, one rarely needs to determine the rational functions explicitly, even on this covering.

When studying the structure sheaf, one always works with affine open sets.

- *Information about an open set that isn't affine is determined by the sheaf property.*

intrstrdef **3.6.6. Definition.** The *structure* of a projective variety X consists the topological space X, together with the structure sheaf $\mathcal{O}_X$.

sectmorph 3.7 Morphisms and Isomorphisms

We have defined morphisms between affine varieties before. If $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$, morphisms $Y \to X$ corresponds to algebra homomorphisms $A \to B$. Moveover, if $A = \mathbb{C}[x]/(f)$, such a homomorphism is determined by a solution of the equations $f = 0$ in B. But it is clear from Example 3.6.1 that one cannot define morphisms of projective varieties directly in terms of the projective coordinates.

Recall that if $T \xrightarrow{u} S$ is a map of sets and g is a function on S, composition with u produces a function $g \circ u = u^* g$ on T, the *pullback* of g.

defmorphism **3.7.1. Definition.** Let X and Y be varieties. A *morphism* $Y \xrightarrow{u} X$ is a continuous map with this property: Let q be a point of Y. If p is the image of q in X, and if f is a rational function on X that is regular at p, its pullback $u^* f$ is a regular function at q.

An *isomorphism* of varieties $Y \xrightarrow{u} X$ is a bijective morphism whose inverse function is also a morphism. For example, the Veronese map (**3.3.3**) defines an isomorphism onto its image. The composition of morphisms is a morphism.

morphtoline **3.7.2. Proposition.** *Morphisms from an arbitrary variety Y to the affine line $\mathbb{A}^1 = \operatorname{Spec} \mathbb{C}[x]$ correspond bijectively to global sections of the structure sheaf $\mathcal{O}_Y$ on Y.*

proof. Let $Y \xrightarrow{u} \mathbb{A}^1$ be a morphism. Since the coordinate function x on $\mathbb{A}^1$ is regular, its pullback $f = u^*x$ is a regular function on Y, a global section of $\mathcal{O}_Y$. This pullback f is the global section that corresponds to u. If a point q of Y has image a in $\mathbb{A}^1$, i.e., if $u(q)$ is the point $x = a$, then $f(y) = a$.

Conversely, if f is a global section of $\mathcal{O}_Y$, we define a map $Y \xrightarrow{u} \mathbb{A}^1$ as follows: Given a point q of Y, we define $u(q)$ to be the point $x = a$, where $a = f(q)$. Then $u^*x(q) = x(u(q)) = a$. So u^*x is the regular function f. Then the pullback of any polynomial $g(x)$ will be regular too. The regular functions on $\mathbb{A}^1$ are the polynomial functions, so the pullback of a regular function on $\mathbb{A}^1$ is regular on Y. It follows that u is a morphism. □

Xmaps **3.7.3. Corollary.** *Let Y be a variety, and let A be a finite-type algebra whose elements are regular functions on Y, global sections of $\mathcal{O}_Y$. There is a canonical morphism $Y \to \operatorname{Spec} A$.*

proof. We choose a finite set $\alpha_1, ..., \alpha_k$ of algebra generators for A, so that $A \approx \mathbb{C}[x_1, ..., x_k]/P$ for some prime ideal P. The generators define morphisms $Y \xrightarrow{u_i} \mathbb{A}^1$ as in the previous proposition, and therefore a morphism $Y \xrightarrow{v} \mathbb{A}^k$. Since the functions in Y defined by elements of P are zero, the image of v is contained in the locus $V(P)$, which is isomorphic to $\operatorname{Spec} A$. □

The definition of morphism can be stated in another way, in terms of the structure sheaf. A continuous map $Y \xrightarrow{u} Y$ is a morphism if it has this property:

Let f be a regular function on an open subset U of X. The pullback u^*f of f is a regular function on $V = u^{-1}(U)$. Or, if $f \in \mathcal{O}_X(U)$, then $f \circ u \in \mathcal{O}_Y(V)$.

Thus, u is a morphism if for every open subset U of X the pullback defines an algebra homomorphism $\mathcal{O}_Y(V) \xleftarrow{u^*} \mathcal{O}_X(U)$.

opensubar

(3.7.4) open subvarieties

The structure sheaf allows us to define a structure of variety on a nonempty open subset X' of a projective variety X. As topological space, we give X' the subspace topology: A subset of X' is open in X' if it is open in X. Then the structure sheaf $\mathcal{O}_{X'}$ on X' is simply the restriction of the structure sheaf on X. A rational function α is regular on an open subset V' of X' if it is regular when V' is viewed as an open subset of X:

$$\mathcal{O}_{X'}(V') = \mathcal{O}_X(V'), \quad \text{if } V' \text{ is a open in } X'.$$

A nonempty open subset X' of a projective variety, together with its structure sheaf $\mathcal{O}_{X'}$, is called a *quasiprojective variety*. Affine and projective varieties are quasiprojective.

affinequasiproj **3.7.5. Proposition.** *An irreducible closed subset of a quasiprojective variety is a quasiprojective variety.*

subsetchain **3.7.6. Lemma.** *Let X be a topological space, let X' be an open subset of X, and let C' be an irreducible closed subset of X'. Let C be the closure of C' in X. Then C is irreducible, and $C' = C \cap X'$.*

proof. Suppose that $C = Z_1 \cup Z_2$, where Z_i are closed subsets of X. Then $Z_i' = Z_i \cap X'$ is closed in X' and $Z_1' \cup Z_2' = C'$. Since C' is irreducible, it is equal to one of these subsets, say $C' = Z_1'$. Then C is the closure $\overline{Z}_1'$ of Z_1' in X, and since Z_1 is closed, $\overline{Z}_1' \subset Z_1$. So $C = \overline{Z}_1' \subset Z_1 \subset C$, and therefore $C = Z_1$. This shows that C is irreducible. The complement V' of C' in X' is open in X', and therefore open in X. The complement of V' in X is a closed subset Y of X that contains C. Then $C' \subset C \cap X' \subset Y \cap X' = C'$, so $C' = C \cap X'$. □

proof of Proposition 3.7.5. Say that Y' is an irreducible closed subset of a quasiprojective variety X' and that X' is an open subvariety of a projective variety X. The closure Y of Y' in X is an irreducible closed subset of X and of the projective space that contains X, so it is a projective variety, and since $Y' = Y \cap X$, Y' is an open subvariety of Y. □

varisquasiproj **3.7.7. Terminology.** We work almost exclusively with quasiprojective varieties in these notes. Since the word "quasiprojective" is ugly, and in order to simplify terminology, we will henceforth use the word "variety" to mean "quasiprojective variety", unless the contrary is stated explicitly.

One part of the next lemma explains how to verify that a map of varieties $Y \xrightarrow{u} X$ is a morphism or an isomorphism, without looking at all open sets.

restrictmorphism **3.7.8. Lemma.** *Let $Y \xrightarrow{u} X$ be a set-theoretic map between varieties, and let Z be an open subvariety or a closed subvariety of X.*
(i) *The inclusion of Z into X is a morphism.*
(ii) *If $\{Y_j'\}$ is a covering of Y by open subvarieties, then u is a morphism if and only if the restricted maps $Y_j' \to X$ are morphisms.*
(iii) *If the image of Y is contained in Z, then u is a morphism if and only if its restriction to a map $Y \to Z$ is a morphism.* □

proof. **(iii)** The assertion is elementary when Z is open in X, so we assume that Z is closed in X, and that Z contains the image of Y. Let $Y \xrightarrow{u_Z} Z$ be the restricted map. Then u is the composition of u_Z with the inclusion $Z \to X$. So u is a morphism if u_Z is a morphism. Suppose that u is a morphism. The map u_Z is continuous because the topology on Z is induced from the topology on X. To show that u_Z is a morphism, we choose affine open subsets X_i of X and Y_j of Y such that u maps each Y_j to some X_i, say with $i = i(j)$, and such that the open sets Y_j cover Y. Then $Z_i = Z \cap X_i$ is a closed subvariety of the affine variety X_i, so it is an affine variety. According to **(ii)**, it suffices to show that the restricted maps $Y_j \longrightarrow Z_i$ are morphisms. This means that it suffices to prove the assertion in the case that Y, X, and therefore Z, are affine varieties.

Say that $Y = \operatorname{Spec} B$, $X = \operatorname{Spec} A$ and that Z is the zero set of the prime ideal P of A. So $Z = \operatorname{Spec} A/P$. The morphism u corresponds to an algebra homomorphism $A \xrightarrow{\varphi} B$. If α is an element of P, then α vanishes on Z. Since the image of Y is contained in Z, $\varphi(\alpha) = 0$. So P is contained in the kernel of φ, and therefore φ induces a homomorphism $A/P \to B$ that defines the restricted morphism u_Z. □.

There is something that we must do. That is to reconcile the definition of morphism given here with the one given for affine varieties in Chapter **??**.

morphmorph **3.7.9. Proposition.** *Let Y and X be affine varieties $\operatorname{Spec} B$ and $\operatorname{Spec} A$, respectively. Morphisms $Y \xrightarrow{u} X$, as defined in (3.7.1), correspond bijectively to algebra homomorphisms $A \xrightarrow{\varphi} B$.*

proof. Let $Y \xrightarrow{u} X$ be a morphism. Since $A = \mathcal{O}_X(X)$, and $B = \mathcal{O}_X(Y)$, the pullback sends elements of A to B, and defines an algebra homomorphism. Conversely, let $A \xrightarrow{\varphi} B$ be an algebra homomorphism. We have seen that φ induces a continuous map $Y \xrightarrow{u} X$ (**??**). Suppose that a rational function f on X is regular at $p = u(q)$. We show that its pullback is regular at q. Since f is regular at p, there will be a (simple) localization $X_s = \operatorname{Spec} A_s$ that contains p, such that f is in A_s. Let $\overline{s}$ denote the image of s in B. This is a nonzero element of B because $\overline{s}(q) = s(p) \neq 0$. The inverse image of X_s is the localization $Y_{\overline{s}} = \operatorname{Spec} B_{\overline{s}}$ of Y, the set of points of Y at which $\overline{s}$ doesn't vanish. This localization contains q, and φ induces a map $A_s \to B_{\overline{s}}$ that sends f to u^*f. So u^*f is in $B_{\overline{s}}$, and is therefore regular at q. □

affopens **(3.7.10) affine open subvarieties**

Since we now have the concepts of open subvarieties and isomorphisms, we can define affine open subvarieties of a variety. An open subset U of a variety X is an *affine open* if there is an affine variety $U' = \operatorname{Spec} A$ and an isomorphism of varieties $U' \to U$.

This gives us a definition, but it doesn't tell us how to decide whether or not a given open set is affine. Indeed, it may be difficult to decide this. Fortunately, there are enough affine open sets that we do know, the special affine open sets that were defined in Section 3.5.

mapprod

3.8 Digression: Mapping Properties

We have been using products of varieties, but we haven't discussed their structure as varieties. The product $\mathbb{P}^m \times \mathbb{P}^n$ of projective spaces can be made into a projective variety by identifying it with its Segre image Π (**3.3.11**). This provides a definition, but one that seems rather arbitrary. The clearest way to describe the product $X \times Y$ of two varieties is in terms of its mapping property. We discuss mapping properties here.

Let $\mathcal{C}$ be any category. A (contravariant) *functor*

$$F:\ \mathcal{C}^\circ \longrightarrow \text{(sets)}$$

sends objects to sets and morphisms to maps of sets, with the direction of the arrows reversed.

Let (T, Z) denote the set of morphisms $T \to Z$ in $\mathcal{C}$. The *mapping property* of an object Z is the functor $F(\,\cdot\,) = (\,\cdot\,, Z)$ that sends a variable "test" object T to the set (T, Z) of morphisms $T \to Z$, and that sends a morphism $T \xrightarrow{f} U$ of objects to the map of sets (in the opposite direction) $(T, Z) \xleftarrow{\circ f} (U, Z)$ that is obtained by composition with f:

mappropZ (3.8.1)

$$\begin{array}{ccc} T & \xrightarrow{u\circ f} & Z \\ {\scriptstyle f}\big\downarrow & & \big\| \\ U & \xrightarrow{u} & Z \end{array}$$

The mapping property of Z is also called the *functor represented by* Z.

The *Yoneda Lemma*, which is proved below, asserts that an object Z of a category is determined up to unique isomorphism by its mapping property. Thus a variety Z has a *mapping property*, the functor $(\,\cdot\,, Z)$

homtovariety (3.8.2)

$$\textit{(varieties)}^\circ \to \text{(sets)},$$

that describes the variety up to unique isomorphism.

We now can describe a variety in either of two ways: as a topological space X with its structure sheaf $\mathcal{O}_X$, or by its mapping property.

Lemma 3.7.8 **(iii)** describes the mapping property of a subvariety Z of a variety X in terms of the mapping property of X:

mapping-propopen **3.8.3. Corollary.** *Let Z be an open or a closed subvariety of a variety X. Morphisms $T \to Z$ correspond bijectively to morphisms $T \to X$ whose images are contained in Z.* $\square$

The next lemma is a more precise version of part **(ii)** of Lemma 3.7.8. It tells us that morphisms $T \to Z$ are determined by their restrictions to an arbitrary open covering of T.

restrtoopentwo **3.8.4. Lemma.** *Let T and Z be varieties, and let $\{T_i\}$ be an open covering of T. Morphisms $T \xrightarrow{f} Z$ correspond bijectively to families $\{T_i \xrightarrow{f_i} Z\}$ of morphisms such that the restrictions of f_i and f_j to $T_i \cap T_j$ are equal.*

proof. A set of morphisms f_i that agree on $T_i \cap T_j$ does define a set-theoretic map $T \xrightarrow{f} Z$, and Lemma 3.7.8**(ii)** asserts that f is a morphism. $\square$

morphtopone **3.8.5. Example.** A morphism $T \xrightarrow{f} \mathbb{P}^1$ to the projective line is determined by an open covering $T^0 \cup T^1$ of T, and sections s_0, s_1 of the structure sheaf $\mathcal{O}_T$ on T^0, T^1, respectively, such that $s_1 = {s_0}^{-1}$ on $T^0 \cap T^1$.

To verify this, we let $U^0 = \operatorname{Spec} A_0$ and $U^1 = \operatorname{Spec} A_1$ denote the standard affine open subsets of $\mathbb{P}^1$. So $A_0 = \mathbb{C}[t]$ and $A_1 = \mathbb{C}[t^{-1}]$. If $f : T \to \mathbb{P}^1$ is a morphism, the inverse images of U^0 and U^1 form an open covering T^0, T^1 of T, and the pullback of t is a section s_0 of $\mathcal{O}_T$ on T^0, and the pullback of t^{-1} is a section s_1 of $\mathcal{O}_T$ on T^1. Conversely, morphisms $f_i : T^i \to U^0$ are defined by such sections. The condition $s_1 = {s_0}^{-1}$ tells us that $f_0 = f_1$ on $T^0 \cap T^1$. $\square$

To state the Yoneda Lemma, we need the concept of an *isomorphism of functors*. Let F and G be functors $\mathcal{C}^\circ \to$ (sets). An isomorphism $F \xrightarrow{\theta} G$ consists of bijective maps $F(T) \xrightarrow{\theta_T} G(T)$ for every object T, compatible with morphisms in the category. So if $T \xrightarrow{f} U$ is a morphism in the category $\mathcal{C}$, the diagram of set maps below commutes

isofnctr (3.8.6)

$$\begin{array}{ccc} F(T) & \xrightarrow{\theta_T} & G(T) \\ {\scriptstyle F(f)}\big\uparrow & & \big\uparrow{\scriptstyle G(f)} \\ F(U) & \xrightarrow{\theta_U} & G(U) \end{array}$$

yoneda **3.8.7. Theorem. (Yoneda Lemma)** *An element Z of a category $\mathcal{C}$ is determined up to unique isomorphism by its mapping property $(\,\cdot\,,Z)$. More precisely, an isomorphism of functors $(\,\cdot\,,Y)\xrightarrow{\theta}(\,\cdot\,,Z)$ determines an isomorphism $Y\xrightarrow{\epsilon}Z$.*

proof. The isomorphism θ gives us bijective maps $(T,Y)\xrightarrow{\theta_T}(T,Z)$ for every T. Setting $T=Y$, we obtain a bijective map $(Y,Y)\xrightarrow{\theta_Y}(Y,Z)$. Via this map, the identity morphism id_Y is sent to a morphism $Y\to Z$ that we denote by ϵ. This will be the isomorphism determined by θ.

For the isomorphism θ and the morphism ϵ, the diagram (3.8.6) becomes

yyzz (3.8.8)
$$\begin{array}{ccc}(Y,Y) & \xrightarrow{\theta_Y} & (Y,Z)\\ {\scriptstyle\circ\epsilon}\uparrow & & \uparrow{\scriptstyle\circ\epsilon}\\ (Z,Y) & \xrightarrow{\theta_Z} & (Z,Z)\end{array}$$

We also have a bijection $(Z,Y)\xrightarrow{\theta_Z}(Z,Z)$. Let $\delta:Z\to Y$ be the inverse image of the identity map id_Z. This will be the inverse of ϵ.

The maps in the above diagram act in this way on the morphisms ϵ and δ:

deltaepsilon (3.8.9)
$$\begin{array}{ccc}\delta\circ\epsilon & \longrightarrow & \epsilon\\ \uparrow & & \uparrow\\ \delta & \longrightarrow & id_Z\end{array}$$

So the image of $\delta\circ\epsilon$ in ϵ in (Y,Z), the same as the image of id_Y. Since θ_Y is bijective, $\delta\circ\epsilon=id_Y$. Replacing θ by its inverse, one shows in the same way that $\epsilon\circ\delta=id_Z$. So ϵ is an isomorphism. □

prodvar 3.9 Product Varieties

Products of affine varieties were described in the previous chapter. If $X=\operatorname{Spec}A$ and $Y=\operatorname{Spec}B$ are affine varieties whose coordinate rings are presented as $A=\mathbb{C}[x]/(\widetilde{f})$ and $B=\mathbb{C}[y]/(\widetilde{g})$, the product variety $X\times Y$ is the affine variety whose coordinate ring is the tensor product algebra

prodaffine (3.9.1)
$$A\otimes_{\mathbb{C}}B=\mathbb{C}[x,y]/(\widetilde{f}(x),\widetilde{g}(y)).$$

We describe a product of arbitrary varieties by its mapping property here.

mapprod **3.9.2. Theorem. (mapping property of the product)** *Let X and Y be varieties. The product set $X\times Y$ has a unique structure of variety with these properties:*
(i) *The projections $X\times Y\xrightarrow{\pi_1}X$ and $X\times Y\xrightarrow{\pi_2}Y$ are morphisms.*
(ii) *For any variety T, the map $(T,X\times Y)\to(T,X)\times(T,Y)$ that sends a morphism h to the pair (π_1h,π_2h) is bijective.*
Thus morphisms $T\xrightarrow{h}X\times Y$ correspond to pairs (r,s) of morphisms $T\xrightarrow{r}X$ and $T\xrightarrow{s}Y$:

proof. The uniqueness of the structure in this theorem follows from the Yoneda Lemma. Any representation of the product $X\times Y$ as a variety that has that mapping property determines its structure uniquely.

For the product of affine varieties, the mapping property follows from Proposition , which tells us that for any algebra S,
$$\operatorname{Hom}(A\otimes B,S)\approx\operatorname{Hom}(A,S)\times\operatorname{Hom}(B,S).$$
Indeed, Lemma 3.8.4 shows that to verify the mapping property, it suffices to consider the case that T is affine, say $T=\operatorname{Spec}R$. Then morphisms $T\to X\times Y$ correspond to algebra homomorphisms $A\otimes B\to R$.

We consider the product $\mathbb{P}^m\times\mathbb{P}^n$ of projective spaces next. As we have mentioned, $\mathbb{P}^m\times\mathbb{P}^n$ can be realized as a projective variety by its Segre embedding Π. Its structure sheaf is determined by that embedding. We describe the sections of the structure sheaf of Π on the affine open set $\Pi^{00} = \Pi \cap W^{00}$, where W^{00} is the standard affine open set $\{w_{00} \neq 0\}$ of $\mathbb{P}^N$. We set $w_{00} = 1$, obtaining the equations $w_{ij} = w_{i0}w_{0j}$ that define Π^{00}, as in (3.3.14). These equations allow us to eliminate the variables w_{ij}, when $i, j > 0$. So the coordinate algebra of Π^{00} is a polynomial ring in the variables w_{i0} and w_{0j}, with $i, j > 0$.

Let U^0 and V^0 denote the standard affine open sets $\{x_0 \neq 0\}$ and $\{y_0 \neq 0\}$ in $\mathbb{P}^m_x$ and $\mathbb{P}^n_y$, respectively. A point w of Π^{00} is the image of a point (x, y) of $U^0 \times V^0$, with $x_i = w_{i0}$, $y_j = w_{0j}$, and $w_{00} = x_0 = y_0 = 1$. This identifies Π^{00} as the affine space $\mathbb{A}^{m+n}$ with coordinates x, y. The result is as expected: $\Pi = \mathbb{P}^m \times \mathbb{P}^n$ is covered by the sets W^{ij}, and W^{ij} is isomorphic to the product $U^i \times V^j$ of standard affine spaces.

Lemma 3.7.8 **(ii)** shows that the two projections $\Pi \to \mathbb{P}^n$ are morphisms. A pair of morphisms $T \to \mathbb{P}^m$ and $T \to \mathbb{P}^n$ gives us a set-theoretic map $T \xrightarrow{h} \mathbb{P}^m \times \mathbb{P}^n$. Let T^{ij} be the inverse image of $U^i \times U^j$. The restriction of h to a map $T^{ij} \to \Pi^{ij} \approx U^i \times U^j$ is a morphism (this is the affine case). Lemma 3.8.4 shows that h is a morphism. Thus the mapping property is true, and $\Pi = \mathbb{P}^m \times \mathbb{P}^n$.

Next, let $X \subset \mathbb{P}^m$ and $Y \subset \mathbb{P}^n$ be closed subvarieties. Then $X \times Y$ is closed in $\mathbb{P}^m \times \mathbb{P}^n = \Pi$ (see Proposition 3.3.16), and therefore

$$(T, X \times Y) = \{h \in (T, \Pi) \mid \text{image of } h \text{ is in } X \times Y\}$$

It follows that $X \times Y$ has the mapping property of the product.

Similar reasoning shows that if X' and Y' are open subvarieties of X and Y, respectively, the open subset $X' \times Y'$ of $X \times Y$ is the product variety. So if X' and Y' are affine open subvarieties, then $X' \times Y'$ is affine too. □

Next comes an important fact about affine open sets that I find surprising, because we know rather little about which open sets are affine.

intersectaffine **3.9.3.** ***Theorem.*** *The intersection $U \cap V$ of affine open subsets U and V of a quasiprojective variety X is an affine open subset of X.*

opencap-closed **3.9.4. Lemma.** *Let $U \xrightarrow{u} X$ and $Y \xrightarrow{y} X$ be inclusions of an open subvariety U and a closed subvariety Y into a variety X. The intersection $Y \cap U = Z$ is an affine variety, a closed subvariety of U and an open subvariety of Y.*

proof. The intersection $Y \cap U$ is an irreducible closed subset of the affine variety U, so it is affine (3.7.6). □

proof Theorem 3.9.3. Let U and V be affine open subvarieties of a quasiprojective variety X. They will also be open subsets of the closure $\overline{X}$ of X in projective space. We may replace X by $\overline{X}$, so we may assume that X is projective. We note that $U \times V$ is an affine open subset of $X \times X$.

Let W be the intersection of $U \times V$ with the diagonal X_Δ in $X \times X$. We form a diagram in which the maps are inclusions:

deltadiagram (3.9.5)
$$\begin{array}{ccc} W & \longrightarrow & U \times V \\ \downarrow & & \downarrow \\ X_\Delta & \longrightarrow & X \times X \end{array}$$

Since $U \times V$ is an affine open subvariety of $X \times X$ and X_Δ is a closed subvariety of $X \times X$, Lemma 3.9.4 tells us that W is affine. When W is regarded as a subset of X via the isomorphism of X with X_Δ, it is the intersection $U \cap V$. So $U \cap V$ is affine. □

abstractva-rieties 3.10 Abstract Varieties

Let X be a topological space. Any (contraviariant) functor

functortoalg (3.10.1)
$$\textit{(opens)}^\circ \xrightarrow{\mathcal{O}} \textit{(algebras)}$$

from open sets to algebras that has the sheaf property, whether or not it is the structure sheaf of a projective variety, can be restricted to an open subset, and it makes sense to say that, with the restricted sheaf, an open subset becomes an affine variety. This allows us to define the concept of an abstract variety. An *abstract variety* is a pair $(X, \mathcal{O})$ of an irreducible topological space and a sheaf $\mathcal{O}$ on X, such that X has an open covering by affine varieties.

An abstract variety is a quasiprojective variety if it is isomorphic to an open subvariety of a projective variety. Neither Proposition 3.3.8, that the diagonal is a closed subset of $X \times X$, nor Theorem 3.9.3, that the intersection of affine open sets is affine, are true for abstract varieties.

Since projective varieties are the most important ones, one would hope that they could be described simply, but the properties that characterize projective varieties aren't well understood. The best charactization known is a result of Kleiman and Benoist. It uses the concept of normal variety, which is given in Chapter **??**.

finitesubset-projective

3.10.2. *Theorem.* *A normal abstract variety is quasiprojective if and only if every finite subset is contained in an affine open set.* □

pointsinfield

3.11 Points with Values in a Field

Let K be a field that contains the complex numbers, and let $X = \operatorname{Spec} A$ be an affine variety. A *point of X with values in K* is an algebra homomorphism $A \to K$. We may denote such a point by $\widetilde{p}$, using the tilde to help us remember that this isn't an ordinary point – a point with values in $\mathbb{C}$.

The substitution principle tells us that algebra homomorphisms $\mathbb{C}[x_1, ..., x_n] \to K$ are given by assigning the images a_i of x_i arbitrarily. So points of affine space $\mathbb{A}^n$ with values in K correspond to arbitrary vectors $(a_1, ..., a_n)$ with entries in K. If $A = \mathbb{C}[x_1, ..., x_n]/(f_1, ..., f_k)$, a point of X with values in K is given by a solution of the equations $f = 0$ in K.

One defines a point of X with values in any algebra R in the same way, as an algebra homomorphism $A \to R$, or as a solution of the system of equations $f(x) = 0$ in R. For example, a point of X with values in $\mathbb{C}[t]$ is given by polynomials $x_i(t)$, $i = 1, ..., n$, such that $f(x(t)) = 0$. Such a point may be thought of as a *polynomial path* in X.

Points of projective space with values in a field K are defined similarly, as equivalence classes of nonzero vectors $(\alpha_0, ..., \alpha_n)$ with α_i in K^n. The equivalence relation is that $(\alpha_0, ..., \alpha_n) \sim (\lambda\alpha_0, ..., \lambda\alpha_n)$ for any nonzero element λ of K. As with points with values in $\mathbb{C}$, one can represent the point (α) in a unique way, with α_0 normalized to 1, provided that α_0 isn't zero.

If X is a projective variety, say $X \subset \mathbb{P}^n$, a point of X with values in K is a point of $\mathbb{P}^n$ with values in K, that solves the homogeneous polynomial equations that define X, i.e., that "lies on X". But for points with values in an algebra R that isn't a field, the equivalence relation may cause trouble. One has to be more careful.

Let K be a field and let (α) be a point of projective space $\mathbb{P}^n$ with values in K. Let P be the homogeneous ideal of $\mathbb{C}[x_0, ..., x_n]$ generated by the homogeneous polynomials $f(x_0, ..., x_n)$ such that $f(\alpha) = 0$. It is easily seen that P is a prime ideal (see (**??**)). The subvariety $Z = V(P)$ of $\mathbb{P}^n$ is called the *Zariski closure* of the point (α).

##Put Z closure of point of $\mathbb{A}^n$ with vaues in K.##

intersectz-closure

3.11.1. Proposition. *With notation as above, if $\alpha_0 = 0$, Z is contained in the hyperplane $H : \{x_0 = 0\}$. If $\alpha_0 \neq 0$, the intersection $Z^0 = Z \cap U^0$ of the Zariski closure Z of (α) with the standard affine open subset U^0 of $\mathbb{P}^n$ is the affine variety defined by the dehomogenizations $f(1, u_1, ..., u_n)$ of the homogeneous elements of the ideal P.*

proof. The assertion when $\alpha_0 = 0$ is obvious. Suppose that $\alpha_0 \neq 0$. Then the point (α) is represented uniquely by the vector $(1, \gamma_1, ..., \gamma_n)$, where $\gamma_i = \alpha_i/\alpha_0$. If f is a homogeneous polynomial suc that $f(\alpha) = 0$, then $f(1, \gamma) = 0$, and conversely, if $F(u_1, ..., u_n)$ is a polynomial such that $F(\gamma) = 0$ and if f is its homogenization. then $f(\alpha) = 0$. □

dimatmosttd

3.11.2. Proposition. *Let Z be the Zariski closure of a point with values in a field K. The dimension of Z is at most equal to the transcendence degree of K over $\mathbb{C}$.*

proof. This is true because the dimension of the affine variety Z^0 is equal to the transcendence degree of its fraction field, which will be a subfield of K. □

morphprojspace

3.12 Morphisms to Projective Space I

We analyze morphisms from an arbitrary variety Y to projective space. To begin, let's suppose that Y is affine, say $Y = \operatorname{Spec} B$. Morphisms from Y to affine space $\mathbb{A}^n$ are given by n-tuples $\beta = (\beta_1, ..., \beta_n)$ of elements of B. The morphism determined by β sends a point p of Y to the point $\beta(p)$ of $\mathbb{A}^n$ obtained by evaluation at p. However, a morphism from Y to projective space needn't be determined by a point $\alpha = (\alpha_0, ..., \alpha_n)$ with values in B. To define a map, one must evaluate α, and evaluation at a point p determines a point of $\mathbb{P}^n$ only if $\alpha_i(p)$ aren't all zero. If for every point p of Y, there is an i such that $\alpha_i(p) \neq 0$, i.e., if the intersection of the zero sets of α_i in Y is empty, then α does define a morphism to $\mathbb{P}^n$. But there may be morphisms that cannot be defined by a single vector α.

maptoP **3.12.1. Example.** Let Y be the cusp curve $\operatorname{Spec} B$, where $B = \mathbb{C}[x,y]/(y^2 - x^3)$. We describe a map $Y \to \mathbb{P}^1$. The algebra B embeds as subring into $\mathbb{C}[t]$, by

$$x = t^2. \;\; y = t^3.$$

This gives us a map from the affine line $\operatorname{Spec} \mathbb{C}[t]$ to Y. Moreover, because $t = y/x$ and $t^{-1} = y/x^2$, $B[x^{-1}]$ is isomorphic to $\mathbb{C}[t, t^{-1}]$. The fraction field K of $\mathbb{C}[t]$ is equal to that of B.

We define a morphism $Y \to \mathbb{P}^1$ using the two vectors $v_0 = (x-1, y-1)$ and $v_1 = (t+1, t^2+t+1)$. These vectors define the same point of $\mathbb{P}^1$ with values in K because $v_0 = (t-1)v_1$. Because the vector v_0 has entries in B, it defines a morphism to $\mathbb{P}^1$ wherever the two entries aren't both zero. The relation $y^2 = x^3$ shows that if $x = 1$, then $y = 1$ too, so v_0 defines a morphism from the complement Y^0 of the point $(1,1)$ to $\mathbb{P}^1$.

Next, the entries of v_1 aren't both zero anywhere. So v_1 defines a morphism from Y to $\mathbb{P}^1$ wherever its entries are regular functions on Y. Since $t = y/x$, this includes all points where $x \neq 0$, which is to say, all points except the point $(0,0)$ of Y. So v_1 defines a morphism from the complement Y^1 of the origin in Y to $\mathbb{P}^1$. These two morphisms piece together to give us a morphism $Y \to \mathbb{P}^1$. □

morphtop **3.12.2. Proposition.** *Let X be a variety with function field K.*

(i) *A morphism $X \xrightarrow{f} \mathbb{P}^n$ from X to projective space determines a point of $\mathbb{P}^n$ with values in K.*

(ii) *Let $(\beta_0, ..., \beta_n)$ be a point of $\mathbb{P}^n$ with values in K, and let V^i denote the largest open subset of X on which all of the functions $\beta_0/\beta_i, ..., \beta_n/\beta_i$ are regular. The point β determines a morphism $X \to \mathbb{P}^n$ if and only if $X = \bigcup V^i$.*

proof. **(ii)** We write the point with values in K as $(\gamma_0, ..., \gamma_n)$, where $\gamma_j = \beta_j/\beta_i$ and $\gamma_i = 1$. The functions γ_j are regular on V^i, and they define a morphism from V^i to the standard affine open U^i of $\mathbb{P}^n$. Piecing these morphisms together gives us the required morphism $X \to \mathbb{P}^n$ (see 3.7.8). □