

Chapter 1 PLANE CURVES

- 1.1 The Affine Plane
- 1.2 The Projective Plane
- 1.3 Loci in Projective Space
- 1.4 Tangent Lines
- 1.5 Resultants and Discriminants
- 1.6 Curves as Coverings of the Projective Line
- 1.7 Bézout's Theorem
- 1.8 Genus
- 1.9 Projective Space is Proper
- 1.10 Nodes and Cusps
- 1.11 Transcendence Degree
- 1.12 The Dual Curve

Plane curves were the first algebraic varieties to be studied in detail, and they provide examples that will be helpful for the understanding of higher dimensional varieties. So we begin with them. We defer proofs of some theorems to later chapters.

affineplane

1.1 The Affine Plane

The n -dimensional *affine space* \mathbb{A}^n is the space of n -tuples of complex numbers. The *affine plane* is the 2-dimensional affine space.

Let $f(x_1, x_2)$ be an irreducible polynomial in two variables, with complex coefficients. The set X of points of the affine plane at which f vanishes, the set of *zeros* of f , is an *affine plane curve*.

affcurve

$$(1.1.1) \quad X = \{x \mid f(x) = 0\}.$$

We often use vector notation such as x for (x_1, x_2) .

The *degree* of an affine plane curve is the degree of its irreducible defining polynomial.

figure

polyirred

1.1.2. Note. Most complex polynomials in two or more variables are irreducible – they cannot be factored. This can be shown by a method called “counting constants”. For instance, quadratic polynomials depend on the coefficients of the six monomials in x, y of degree at most two. Linear polynomials $ax+by+c$ depend on three coefficients, but the product of two linear polynomials depends on only five coefficients, because a scalar multiple can be moved from one linear factor to the other. So quadratic polynomials cannot all be written as products of linear polynomials. This reasoning is fairly clear. Its formal justification in terms of *dimension* will be explained in Chapter ??.

We may want to classify curves, and to do this, we must decide when to call curves equivalent. It is customary to allow arbitrary linear changes of variable and translations. If we write x as the column vector $(x_1, x_2)^t$, the coordinates $x' = (x'_1, x'_2)^t$ after such a change of variable will have the form

chgcoord

$$(1.1.3) \quad Px' + a = x,$$

where P is an invertible complex 2×2 matrix and $a = (a_1, a_2)^t$ is a complex translation vector. This changes a polynomial equation $f(x) = 0$ to $f(Px' + a) = 0$. One may also multiply a polynomial by a (nonzero) complex scalar without changing the locus $\{f = 0\}$.

Using these operations, all *affine lines*, curves of degree 1, become equivalent.

Any equation $q(x_1, x_2) = 0$ in which q is an irreducible quadratic polynomial is equivalent to one of the two equations

classconic

$$(1.1.4) \quad x_1^2 - x_2^2 - 1 = 0 \quad \text{or} \quad x_1^2 - x_2 = 0.$$

The loci might be called a complex 'hyperbola' and 'parabola', respectively. The complex 'ellipse' $x_1^2 + x_2^2 - 1 = 0$ becomes equivalent to the hyperbola when one multiplies x_2 by i . The proof that every quadratic equation is equivalent to one of these two is similar to the proof used in the classification of real conics.

On the other hand, there are infinitely many inequivalent affine cubic curves. Cubic polynomials in two variables depend on the coefficients of the ten monomials in x, y of degrees ≤ 3 . Linear operators, translations, and scalar multiplication give us only seven parameters to work with, leaving at least three parameters. In fact, the true number of parameters is three, though two of the three can be eliminated when one uses projective transformations (see Section 1.2.8).

projplane

1.2 The Projective Plane

The n -dimensional *projective space* \mathbb{P}^n is the set of equivalence classes of *nonzero* vectors $x = (x_0, x_1, \dots, x_n)$, the equivalence relation being

$$(x'_0, \dots, x'_n) \sim (x_0, \dots, x_n) \quad \text{if} \quad (x'_0, \dots, x'_n) = (\lambda x_0, \dots, \lambda x_n)$$

for some nonzero complex number λ . The equivalence classes are the *points* of \mathbb{P}^n , and one often refers to a point by giving a particular vector (x_0, \dots, x_n) in its class.

Points of the *projective line* \mathbb{P}^1 are equivalence classes of nonzero vectors (x_0, x_1) . If (x_0, x_1) is such a vector, and x_0 isn't zero, we may multiply by $\lambda = x_0^{-1}$ to normalize the first entry to 1, and write the point it represents in a unique way as $(1, u)$, with $u = x_1/x_0$. There is one remaining point, the point $(0, 1)$. Thus the projective line can be obtained by adding one point, called the *point at infinity*, to the one dimensional complex space that is the affine u -line. Topologically, \mathbb{P}^1 is a two-dimensional sphere.

Let V denote the vector space \mathbb{C}^{n+1} , and let W be a subspace of V , of dimension $r + 1$. The points of \mathbb{P}^n represented by the nonzero vectors in W form a *linear subspace* L of \mathbb{P}^n , of dimension r . If (w_0, \dots, w_r) is a basis of W , the points of L can be written as $p = c_0 w_0 + \dots + c_r w_r$, and the linear subspace L corresponds bijectively to \mathbb{P}^r , by $p \leftrightarrow (c_0, \dots, c_r)$. For example, the set of points $(x_0, \dots, x_r, 0, \dots, 0)$ is a linear subspace of dimension r .

A point x of \mathbb{P}^n is a linear subspace of dimension zero. If x is a nonzero element of V , the vectors (λx) , together with (0) , form the one-dimensional subspace of the complex vector space \mathbb{C}^{n+1} spanned by x . So points of \mathbb{P}^n correspond bijectively to one-dimensional subspaces of \mathbb{C}^{n+1} .

A *line* L in projective space is the one-dimensional linear space defined by a two-dimensional subspace W of V . If $p = (p_0, \dots, p_n)$ and $q = (q_0, \dots, q_n)$ are distinct points of L , the vectors p, q will form a basis for W , and L will be the set $\{rp + sq\}$, with r, s in \mathbb{C} not both zero. The bijective correspondence $L \leftrightarrow \mathbb{P}^1$ defined by this basis is

projline

$$(1.2.1) \quad rp + sq \quad \longleftrightarrow \quad (r, s).$$

In this chapter, we study loci in the *projective plane* \mathbb{P}^2 , the set of equivalence classes of nonzero vectors (x_0, x_1, x_2) . A line in the projective plane can be described, as above, by a pair of points, or as the locus of solutions of a homogeneous linear equation

eqline

$$(1.2.2) \quad c_0 x_0 + c_1 x_1 + c_2 x_2 = 0.$$

linesmeet **1.2.3. Lemma.** *Two distinct lines in the projective plane have exactly one point in common, and a pair of distinct points is contained in exactly one line.* \square

dualplanesect **(1.2.4) the dual plane**

Let \mathbb{P} denote the projective plane with coordinates x_0, x_1, x_2 , and let ℓ be the line whose equation is

lineequation (1.2.5)
$$s_0x_0 + s_1x_1 + s_2x_2 = 0.$$

The solutions of this equation determine the coefficients s_i only up to a nonzero scalar factor, so ℓ determines a point (s_0, s_1, s_2) that we denote by ℓ^* in another projective plane \mathbb{P}^* , the *dual plane*. Moreover, a point (x_0, x_1, x_2) in \mathbb{P} determines a line p^* in the dual plane, the line with the same equation (1.2.5), when s_i are regarded as the variables and x_i as the scalar coefficients. The equation exhibits a duality between \mathbb{P} and \mathbb{P}^* . A point p of \mathbb{P} lies on a line ℓ if and only if the equation is satisfied, and this means that, in \mathbb{P}^* , the line p^* contains the point ℓ^* .

standcov

(1.2.6) the standard affine cover

If $p = (x_0, x_1, x_2)$ is a point of \mathbb{P}^2 , and if $x_0 \neq 0$, we may normalize the first entry to 1 without changing the point: $(x_0, x_1, x_2) \sim (1, u_1, u_2)$, where $u_i = x_i/x_0$. We did this for \mathbb{P}^1 above. The representative vector $(1, u_1, u_2)$ is uniquely determined by p , so points with $x_0 \neq 0$ correspond bijectively to points of an affine plane with coordinates u :

$$(x_0, x_1, x_2) \sim (1, u_1, u_2) \longleftrightarrow (u_1, u_2).$$

We regard the affine plane as a subset of \mathbb{P}^2 by this correspondence, and we denote that subset by U^0 .

When we look at a point of U^0 , we may simply set $x_0 = 1$, and write it as $(1, x_1, x_2)$. To write $u_i = x_i/x_0$ makes sense only when a particular coordinate vector (x_0, x_1, x_2) has been given.

The points of U^0 , those with $x_0 \neq 0$, are the *points at finite distance*. The *points at infinity* of \mathbb{P}^2 , those of the form $(0, x_1, x_2)$, are on the *line at infinity* ℓ^0 , the locus $\{x_0 = 0\}$, which is a projective line. This gives us a dichotomy: \mathbb{P}^2 is the union of the two sets U^0 and ℓ^0 . We can assume that the first coordinate of a given point is either 1 or 0.

There are analogous correspondences between points $(x_0, 1, x_2)$ with $x_1 \neq 0$ and points of an affine plane \mathbb{A}^2 and between points $(x_0, x_1, 1)$ with $x_2 \neq 0$ and points of \mathbb{A}^2 . We denote these subsets by $U^1 : \{x_1 \neq 0\}$ and $U^2 : \{x_2 \neq 0\}$. The three sets U^0, U^1, U^2 form the *standard covering* of \mathbb{P}^2 by three *standard affine open sets*. Since the vector $(0, 0, 0)$ has been ruled out, every point of \mathbb{P}^2 lies in at least one of the standard affine open sets. Points whose three coordinates are nonzero lie in all three.

figure

This discussion extends in a natural way to projective spaces of arbitrary dimension. The subset U^0 of \mathbb{P}^n of points $(1, x_1, \dots, x_n)$ with $x_0 \neq 0$ corresponds bijectively an affine n -space. The *standard affine covering* of \mathbb{P}^n consists of the $n+1$ subsets $U^i : \{x_i \neq 0\}$, each of which is an affine space.

pointatinf-ity **1.2.7. Note.** Which points of \mathbb{P}^2 are at infinity depends on which of the standard affine open sets is taken to be the one at finite distance. When the coordinates are (x_0, x_1, x_2) , I like to normalize x_0 to 1, as above. Then the points at infinity are $(0, x_1, x_2)$. But when coordinates are (x, y, z) , I often normalize z to 1, so that the points at infinity are $(x, y, 0)$. I hope this won't cause too much confusion. \square

A word about figures may be in order. In algebraic geometry, the dimensions of varieties are too big to allow realistic figures. Even with an affine plane curve, one is dealing with a two-dimensional object in the four-dimensional space \mathbb{C}^2 . In some cases, such as with plane curves, depicting the real locus can be helpful. Figures ?? and ?? are examples of this. The real locus can often depict interesting phenomena, but in most cases one must make do with a schematic figure. The figure below is an example. My students tell me that all of my figures look more or less like this:

figure

chgcoordssec

(1.2.8) changing coordinates

An invertible 3×3 matrix P determines a linear change of coordinates in \mathbb{P}^2 . We won't always be careful to mention whether we regard a vector as a row or as a column, but let's use column vectors here. With $x = (x_0, x_1, x_2)^t$ and $x' = (x'_0, x'_1, x'_2)^t$ respectively, the coordinate change is given by

chg (1.2.9)
$$Px' = x.$$

Four special points, the three points $e_0 = (1, 0, 0)$, $e_1 = (0, 1, 0)$, $e_2 = (0, 0, 1)$, and the point $\epsilon = (1, 1, 1)$ determine the coordinates in \mathbb{P}^2 . The first three are connected pairwise by the lines $\{x_0 = 0\}$, $\{x_1 = 0\}$ and $\{x_2 = 0\}$, as in Figure ?? above.

fourpoints

1.2.10. Proposition. *Let p_0, p_1, p_2 , and q be four points of \mathbb{P}^2 , no three of which lie on a line. There is, up to scalar factor, a unique linear change of coordinates $x = Px'$ such that, when $x = p_0, p_1, p_2$, or q , $x' = e_0, e_1, e_2$, or ϵ , respectively.*

proof. We represent the points by specific vectors, and we look for a matrix P such that $Pe_i = p_i$ and $P\epsilon = q$. The statement that the points p_0, p_1, p_2 do not lie on a line means that the vectors are independent. They span \mathbb{C}^3 . So q will be a linear combination $q = c_0p_0 + c_1p_1 + c_2p_2$, and because no three points lie on a line, the coefficients c_i will be nonzero. (If $c_0 = 0$, for instance, then q, p_1, p_2 lie on the line $c_1p_1 + c_2p_2 - q = 0$.)

We can *scale* the vectors p_i (multiply them by nonzero scalars) to make $q = p_0 + p_1 + p_2$. Next, the columns of P can be an arbitrary set of independent vectors. We let them be p_0, p_1, p_2 . Then $Pe_i = p_i$, and $P\epsilon = q$. The matrix P is unique up to scalar factor. This is proved by looking the argument over. \square

projconics

1.2.11. Example: Conics. A polynomial is *homogeneous*, of *degree d* if all monomials that appear with nonzero coefficients have degree d . So a homogeneous quadratic polynomial $f(x_0, x_1, x_2)$ is a combination of the six monomials $x_0^2, x_1^2, x_2^2, x_0x_1, x_1x_2, x_0x_2$. A *conic* in \mathbb{P}^2 is the locus of zeros of an irreducible quadratic polynomial. We show that, using linear coordinate changes, all conics become equivalent.

classifyconic

1.2.12. Proposition. *Every conic is equivalent to the conic $x_0x_1 + x_0x_2 + x_1x_2 = 0$.*

proof. Since a conic isn't a line, it will contain three points that aren't colinear. Let's leave the verification of this fact as an exercise. We choose three non-colinear points, and adjust coordinates so that they are the points e_0, e_1, e_2 . Then $f(1, 0, 0) = 0$, and therefore the coefficient of x_0^2 in f is zero. Similarly, the coefficients of x_1^2 and x_2^2 are zero. So f has the form

$$f = ax_0x_1 + bx_0x_2 + cx_1x_2$$

Since f is irreducible, a, b, c aren't zero. By scaling the variables appropriately, we can make $a = b = c = 1$. We will be left with the polynomial $f_0 = x_0x_1 + x_0x_2 + x_1x_2$. \square

Here is another simple example of a coordinate change: Suppose given a homogeneous polynomial $f(x, y, z)$ of degree d . We write it as a polynomial in z :

polyinztwo

(1.2.13)
$$f(x, y, z) = a_0z^d + a_1z^{d-1} + \dots + a_d$$

where $a_i = a_i(x, y)$ is a homogeneous polynomial of degree i in x, y . We may want f to be monic in z . The coefficient a_0 of z^d in f is a constant that is usually unimportant, provided that it isn't zero. To ensure that z^d has nonzero coefficient, we may choose a point p at which f isn't zero, and adjust coordinates so that p becomes the point $e_2 = (0, 0, 1)$. \square

homogequation

1.3 Loci in Projective Space

The loci in projective space that we will study are defined by systems of *homogeneous* polynomial equations. We require homogeneity because the vectors (a_0, \dots, a_n) and $(\lambda a_0, \dots, \lambda a_n)$ define the same point of \mathbb{P}^n . We

want to know that if $f(x) = 0$ is a polynomial equation, and if $f(a) = 0$, then $f(\lambda a) = 0$ for every $\lambda \neq 0$. As we verify now, and this will be true if and only if f is homogeneous.

A polynomial f can be written as a sum of its *homogeneous parts*:

homparts (1.3.1)
$$f = f_0 + f_1 + \cdots + f_d,$$

where f_0 is the constant term, f_1 is the linear term etc., and d is the degree of f .

hompartzero **1.3.2. Lemma.** *Let f be a polynomial of degree d , and let $x = (x_0, \dots, x_n)$. Then $f(\lambda x) = 0$ for every nonzero complex number λ if and only if $f_i(x) = 0$ for $i = 0, \dots, d$.*

proof. $f(\lambda x_0, \dots, \lambda x_n) = f_0 + \lambda f_1(x) + \lambda^2 f_2(x) + \cdots + \lambda^d f_d(x)$. When we evaluate at a given vector x , the right side of this equation becomes a polynomial of degree at most d in λ . Since a nonzero polynomial of degree $\leq d$ can have no more than d roots, $f(\lambda x)$ will not be zero for every λ unless $f_i(x) = 0$ for every i . \square

multiplyhomogeneous **1.3.3. Lemma.** *A product fg of polynomials is homogeneous if and only if the factors f and g are homogeneous. \square*

We describe subsets of the projective line that are the zero sets of homogeneous polynomials $f(x, y)$ in two variables:

factorhomopoly **1.3.4. Lemma.** *Every homogeneous nonzero polynomial $f(x, y) = a_0x^d + a_1x^{d-1}y + \cdots + a_dy^d$ in two variables, with complex coefficients, is a product of linear polynomials that are unique up to scalar factor. Adjusting scalar factors, we may write*

$$f(x, y) = (v_1x - u_1y)^{r_1} \cdots (v_kx - u_ky)^{r_k}$$

where no factor $v_ix - u_iy$ is a constant multiple of another and where $r_1 + \cdots + r_k$ is the degree d of f . The exponent r_i is called the *multiplicity of the linear factor* $v_1x - u_1y$.

To prove this, one factors the one-variable complex polynomial $f(x, 1)$ into linear factors, substitutes x/y for x , and multiplies the result by y^d . When one adjusts scalar factors, one will obtain the expected factorization of $f(x, y)$. For instance, to factor $f(x, y) = 2x^2 - 3xy + y^2$, we substitute $y = 1$, obtaining $2x^2 - 3x + 1 = 2(x - 1)(x - \frac{1}{2})$. We substitute $x = x/y$ and multiply by y^2 : $f(x, y) = (x - y)(2x - y)$. \square

A linear polynomial $vx - uy$ determines a point (u, v) in the projective line \mathbb{P}^1 , the *zero* of that polynomial, and scalar factors don't change the zero. Thus the linear factors of a homogeneous polynomial f determine points of \mathbb{P}^1 , the *zeros* of f . As with the roots of a one-variable polynomial, one can assign multiplicities to those zeros. When we write f as a product as in Lemma 1.3.4, the points (u_i, v_i) will be distinct zeros of *multiplicity* r_i . The zero (u_i, v_i) of f in \mathbb{P}^1 corresponds to a root $x = u_i/v_i$ of multiplicity r_i of the one-variable polynomial $f(x, 1)$, except when it is the point $(1, 0)$. This happens when $a_0 = 0$, and y is a factor of f . In that case, one may say that $f(x, 1)$ has a root at infinity.

This sums up the information that is contained in an algebraic locus in the projective line. It will be a finite set of points with multiplicities.

projcurve

(1.3.5) plane projective curves

The most interesting loci in the projective plane are the zero sets of single homogeneous polynomial equations $f = 0$. It is reasonable to focus attention on irreducible polynomials because, if g and h are homogeneous polynomials, the locus $\{gh = 0\}$ is the union of the two loci $\{g = 0\}$ and $\{h = 0\}$.

When f is irreducible, the locus $\{f = 0\}$ is a (*projective*) *plane curve*. The *degree* of a plane curve is defined to be the degree of its irreducible defining polynomial.

The zero locus of a reducible polynomial may be called a *reducible curve*. However, when a polynomial f has multiple factors, it may be desirable to keep track of the multiplicities. To do this, one associates an integer combination of curves, called a *divisor*, to f . One writes f as a product of irreducible polynomials, say

factorf (1.3.6)
$$f = \phi_1^{r_1} \cdots \phi_k^{r_k},$$

where ϕ_i are irreducible polynomials and where ϕ_j isn't a scalar multiple of ϕ_i if $i \neq j$. Then if C_i is the plane curve $\{\phi_i = 0\}$, the associated *divisor* is defined to be the integer combination

divisoroff (1.3.7)
$$Z = r_1C_1 + \cdots + r_kC_k$$

The *support* of this divisor is the union of the curves C_i , the reducible curve $\{f = 0\}$, and its *degree* is the sum $r_1d_1 + \cdots + r_kd_k$, where d_i is the degree of C_i . If Z is the divisor of the homogeneous polynomial f , the degree of Z is equal to the degree of f .

zartop

(1.3.8) the Zariski topology

The usual topology on the affine plane \mathbb{A}^2 will be called the *classical topology*. A subset U of \mathbb{A}^2 is open in the classical topology if, whenever U contains a point p , it contains all points sufficiently near to p . We call this the classical topology because the Zariski topology is another topology that we are about to describe.

The projective plane also has a classical topology. A subset U of \mathbb{P}^2 is open if, whenever a point p of U is represented by a vector (x_0, x_1, x_2) , all vectors (x'_0, x'_1, x'_2) with x'_i sufficiently near to x_i represent points of U .

defzartop

1.3.9. Definition. A subset of affine space \mathbb{A}^n is *Zariski closed* if it is the set of common zeros of a family F_1, \dots, F_k of polynomials in n variables x_1, \dots, x_n . A subset of projective space \mathbb{P}^n is *Zariski closed* if it is the set of common zeros of a family of homogeneous polynomials f_1, \dots, f_k in $n + 1$ variables x_0, \dots, x_n .

The proper Zariski closed subsets of the affine or projective line are the nonempty finite subsets. The proper Zariski closed subsets of the affine or projective plane are finite unions of points and curves (??).

pointclosed

1.3.10. Lemma. (i) A subset of \mathbb{P}^n consisting of a single point is Zariski closed.

(ii) Let L be the linear subspace of \mathbb{P}^n defined by a subspace W of \mathbb{C}^{n+1} , and let $L \xrightarrow{u} \mathbb{P}^r$ be the bijective map defined by a basis of W . If Z is a Zariski closed subset of \mathbb{P}^n that is contained in L , its image is a Zariski closed subset of \mathbb{P}^r . ##state iff?? ##

proof. **(i)** This simple proof illustrates a general method. Let p be the point (a_0, \dots, a_n) . The first guess might be that the one-point set $\{p\}$ is defined by the equations $x_i = a_i$. But the polynomials $x_i - a_i$ aren't homogeneous in x . This is reflected in the fact that, for any $\lambda \neq 0$, the vector $(\lambda a_0, \dots, \lambda a_n)$ represents the same point, though it doesn't satisfy those equations. The equations that define the set $\{p\}$ are

definepoint

(1.3.11)
$$a_i x_j = a_j x_i,$$

for $i, j = 0, \dots, n$, which show that the ratios a_i/a_j and x_i/x_j are equal.

(ii) Let $f_\nu(x_0, \dots, x_n)$ be a system of homogeneous polynomials whose zero set is X , and let $w = (w_0, \dots, w_r)$ be a basis of W . Thinking of V as the space \mathbb{C}^{n+1} of column vectors, a linear combination $c_0w_0 + \cdots + c_rw_r$ is a column vector whose coordinates are linear in c . These coordinates can be substituted into f_ν , and via this substitution, $f_\nu(c_0w_0 + \cdots + c_rw_r)$ becomes a homogeneous polynomial in c of the same degree as f_ν . The image is the zero locus of these polynomials. \square

To verify that the Zariski closed subsets of \mathbb{A}^n or \mathbb{P}^n are the closed sets of a topology (the *Zariski topology*), one must show that

- the empty set and the whole space are Zariski closed,
- the intersection $\bigcap C_i$ of an arbitrary family of Zariski closed sets is Zariski closed, and
- the union of two Zariski closed sets is Zariski closed.

We verify this for subsets of projective space. The empty set and the whole space are the zero sets of the polynomials 1 and 0, respectively. If C_i is the zero set of a family S_i of homogeneous polynomials, the intersection $\bigcap C_i$ is the zero set of the union $\bigcup S_i$ of those families. Say that X and Y are the zero sets of the homogeneous polynomials g_1, \dots, g_r and h_1, \dots, h_s , respectively. Then $X \cup Y$ is the zero set of the family of products $\{g_i h_j\}$. Indeed, it is obvious that those polynomials are zero at every point of X and at every point of

Y . Conversely, let p be a point such that $g_i(p)h_j(p) = 0$ for all i, j . If $g_i(p) \neq 0$ for some i , then $h_j(p) = 0$, and this will be true for all j . Therefore p is a point of Y . If $g_i(p) = 0$ for all i , then p is a point of X . In either case, p is a point of $X \cup Y$. \square

Since polynomial functions are continuous, Zariski closed sets are closed in the classical topology as well. The Zariski topology is much coarser than the classical topology. (A *coarser* topology is one with fewer open sets or fewer closed sets. A *finer* topology is one with more open or closed sets.)

Though we will use the classical topology from time to time, the Zariski topology will appear more often. For this reason, we may refer to a Zariski closed subset simply as a *closed set*. Similarly, by an *open set* we will mean a Zariski open set. We will mention the adjective “Zariski” only for emphasis.

The next proposition shows that any two nonempty Zariski open sets have a nonempty intersection. One consequence is that, in the Zariski topology, affine and projective spaces of positive dimension aren't Hausdorff spaces.

opendense

1.3.12. Proposition. *Every nonempty Zariski open subset of \mathbb{A}^n or \mathbb{P}^n is dense and path connected in the classical topology.*

(A subset U of a topological space X is *dense* if its *closure*, the smallest closed set that contains U , is the whole space X .)

proof. We do the case of projective space. Let U be a nonempty Zariski open subset of \mathbb{P}^n , and let X be its closed complement. To show that U is dense in the classical topology, we choose distinct points p and q of \mathbb{P}^n , with p in U . Let ℓ be the line through p and q . According to Lemma 1.3.10(ii), $X \cap \ell$ will be a proper Zariski closed subset of the line ℓ , a finite set. Therefore the closure of $U \cap \ell$ in the classical topology will be the whole line ℓ . It follows that q is in the closure of U , and since q was arbitrary, the closure of U is \mathbb{P}^n .

Next, let ℓ be the line through two points p and q of U . As before, $X \cap \ell$ will be a finite set. We remember that ℓ is a projective line, and that with its classical topology, it is a two-dimensional sphere. So p and q can be joined by a path in ℓ that avoids the finite set $X \cap \ell$. The path can be made as nice as one wants. \square

tanlines

1.4 Tangent Lines

Let C denote the plane curve defined by an irreducible homogeneous polynomial $f(x_0, x_1, x_2)$, and let f_i denote the partial derivative $\frac{\partial f}{\partial x_i}$. A point p of C at which the partial derivatives f_i aren't all zero is called a *smooth point* of C , and a point at which all partial derivatives are zero is a *singular point*.

The meaning of smoothness is explained by the Implicit Function Theorem. Let p be a point of C in the standard affine open set $x_0 \neq 0$. We normalize x_0 to 1 and inspect the locus $f(1, x_1, x_2) = 0$. Let p be a smooth point and suppose that $\frac{\partial f}{\partial x_2} \neq 0$ at p , then we can solve the equation $f(1, x_1, x_2) = 0$ for x_2 locally as an analytic function of x_1 , (The word *locally* means “in some open neighborhood of the point”.)

A curve C is *smooth*, or *nonsingular*, if it contains no singular point; otherwise it is a *singular curve*. The

fermatcurve

$$(1.4.1) \quad x_0^r + x_1^r + x_2^r = 0$$

is smooth because the partial derivatives $rx_0^{r-1}, rx_1^{r-1}, rx_2^{r-1}$ have no common zeros in \mathbb{P}^2 . The cubic curve $x_0^3 + x_1^3 - x_0x_1x_2 = 0$ is singular at the point $(0, 0, 1)$.

We use the same definition for the divisor Z defined by a reducible polynomial f . A *singular point* of Z is one at which all of the partial derivatives f_i are zero.

We will see that a curve can have only finitely many singular points (Corollary 1.6.4). However, if in a divisor $Z = r_1C_1 + \dots + r_kC_k$ some r_i is greater than 1, then every point of C_i will be a singular point of Z . Namely, when we write $f = \phi_1^{r_1} \dots \phi_k^{r_k}$ as in (1.3.6), ϕ_i will divide all partial derivatives of f , so they will vanish on C_i .

eulerformula

1.4.2. Euler's Formula. Let f be a homogeneous polynomial of degree d in the variables x_0, \dots, x_n . Then

$$\sum_i x_i \frac{\partial f}{\partial x_i} = d f.$$

This is checked directly for monomials. For instance, when $f = x^2y^3z$,

$$xf_x + yf_y + zf_z = x(2xy^3z) + y(3x^2y^2z) + z(x^2y^3) = 6x^2y^3z.$$

□

1.4.3. Corollary.

(i) If all partial derivatives of a homogeneous polynomial f are zero at a point p of \mathbb{P}^2 , then f is zero at p , and therefore p lies on the divisor defined by f , and is a singular point of that divisor.

(ii) The partial derivatives of an irreducible polynomial have no common nonconstant factor. □

Let C be the plane curve defined by an irreducible polynomial f , and let p be a smooth point of C . A line ℓ is *tangent* to C at p if the restriction of f to ℓ has a zero of multiplicity at least 2 at p , and p is a *flex point* if the restriction of f to the tangent line ℓ has a zero of multiplicity at least 3 at p .

The tangent line at a smooth point of a curve is unique. To determine it, we work with specific vectors that represent points of the projective plane. Let p be a point of C , let q be a point of \mathbb{P}^2 distinct from p , and let ℓ be the line $\{up + vq\}$. We restrict f to ℓ and expand in a Taylor's series around p , writing a variable point of ℓ as $p + tq$.

taylor (1.4.4)
$$f(p + tq) = f(p) + \left(\sum_i f_i(p) q_i \right) t + \frac{1}{2} \left(\sum_{i,j} q_i f_{ij}(p) q_j \right) t^2 + O(t^3),$$

In this equation, $f(p) = 0$ because p is a point of C , f_i and f_{ij} denote the partial derivatives $\frac{\partial f}{\partial x_i}$ and $\frac{\partial^2 f}{\partial x_i \partial x_j}$, respectively, and $O(t^3)$ stands for a polynomial in t in which all terms have degree at least 3. The point q is missing from this parametrization of ℓ , but this won't be important.

We rewrite the equation in matrix form. Let ∇ denote the gradient vector (f_0, f_1, f_2) , and let H denote the *Hessian matrix*, the symmetric matrix of second partial derivatives of f :

hessianmatrix (1.4.5)
$$H = \begin{pmatrix} f_{00} & f_{01} & f_{02} \\ f_{10} & f_{11} & f_{12} \\ f_{20} & f_{21} & f_{22} \end{pmatrix}$$

Let ∇_p and H_p be the evaluations at p of the gradient and the Hessian matrix, respectively. Regarding p and q as column vectors, Equation 1.4.4 can be written as

texp (1.4.6)
$$f(p + tq) = (\nabla_p q) t + \frac{1}{2} (q^t H_p q) t^2 + O(t^3).$$

The next lemma follows from Euler's formula.

applyeuler **1.4.7. Lemma.** With notation as above, $p^t H_p = (d-1)\nabla_p$ and $p^t H_p p = d(d-1)f(p)$. □

Let's use $\langle x, y \rangle$ to denote the symmetric form $x^t H_p y$ on $\mathbb{C}^3 \times \mathbb{C}^3$. Whether or not $\langle x, y \rangle$ is zero depends only on the points of \mathbb{P}^2 represented by the vectors x and y , so it makes sense to say that this form vanishes on a pair of points.

bilinform **1.4.8. Corollary.** Let p be a smooth point of C and let q be a point of \mathbb{P}^2 distinct from p . Then $\langle p, p \rangle = 0$. The line ℓ defined by (1.4.6) is tangent to C at p if and only if $\langle p, q \rangle = 0$. If $\langle p, q \rangle = 0$, then p is a flex point if and only if $\langle q, q \rangle = 0$. □

Note. Taylor's formula shows that the restriction of f to any line through a singular point has a multiple zero, but we call a line a tangent line only if it is tangent to C at a smooth point.

tangentline **1.4.9. Theorem.** A smooth point p of a curve C is a flex point if and only if the Hessian determinant $\det H_p$ of the Hessian matrix at p is zero.

proof. Let ℓ be the tangent line at a smooth point p , and let q be another point of ℓ . Then $\langle p, p \rangle = \langle p, q \rangle = 0$, from which it follows that p is orthogonal to every point of ℓ . If p is a flex point, then we also have $\langle q, q \rangle = 0$, and therefore q is orthogonal to ℓ too. If the form were nondegenerate, the orthogonal space to ℓ would be a point. This would imply $p = q$, contrary to hypothesis. So the form is degenerate.

Conversely, if the form is degenerate, there will be a null vector q , a vector whose orthogonal space q^\perp is the whole space \mathbb{P}^2 . Since p is a smooth point, the formula $p^t H_p = (d-1)\nabla_p$ shows that p^\perp isn't the whole space. So $q \neq p$. Then the tangent line ℓ is the line through p and q , and since $\langle q, q \rangle = 0$, p is a flex point. □

hessnotzero

1.4.10. Proposition. *Let $f(x_0, x_1, x_2)$ be an irreducible homogeneous polynomial of degree at least 2. The Hessian determinant isn't divisible by f . In particular, it isn't identically zero.*

proof. Let C be the curve defined by f , and suppose that f divides the Hessian determinant. Then every point of C will be a flex point. We set $x_0 = 1$ and look on the standard affine U^0 . At most points p , one can solve the equation $f(1, x_1, x_2) = 0$ locally for x_2 as analytic function of x_1 , say $x_2 = \varphi(x_1)$. If p is a flex point, then $\frac{d^2\varphi}{dx_1^2}$ is zero at p , and if this is true for all points near to p , then φ will be a linear function: $x_2 = ax_1 + b$. Since $x_2 = ax_1 + b$ solves $f = 0$, $x_2 - ax_1 - b$ divides $f(1, x_1, x_2)$. This follows when one divides $f(1, x_1, x_2)$ by the monic polynomial $x_2 - ax_1 - b$ in x_2 . But since $f(x_0, x_1, x_2)$ is irreducible, so is $f(1, x_1, x_2)$. Therefore $f(1, x_1, x_2)$ is linear, and since f has degree at least 2, x_0 divides f , which contradicts irreducibility. \square

resultant

1.5 Resultants and Discriminants

Let F and G be monic polynomials with complex coefficients, say

$$(1.5.1) \quad F(x) = x^m + a_1x^{m-1} + \cdots + a_m, \quad G(x) = x^n + b_1x^{n-1} + \cdots + b_n$$

The resultant $\text{Res}(F, G)$ is a certain polynomial in the coefficients a_i, b_j . Its important property is that, when the coefficients are in a field, the resultant is zero if and only if F and G have a common factor.

The formula for the resultant is nicer when one allows leading coefficients different from 1, but if we do this, we must allow the possibility that the leading coefficient is zero. We work with homogeneous polynomials in two variables to prevent the degree from dropping when that happens.

Let f and g be homogeneous polynomials with complex coefficients, say

$$(1.5.2) \quad f(x, y) = a_0x^m + a_1x^{m-1}y + \cdots + a_my^m, \quad g(x, y) = b_0x^n + b_1x^{n-1}y + \cdots + b_ny^n.$$

If these polynomials have a common zero (u, v) in \mathbb{P}^1 , $vx - uy$ will divide both f and g (see Section 1.3). Then the polynomial $h = fg/(vx - uy)$ will be divisible by f and by g , say $h = pf = qg$, where p and q are homogeneous polynomials of degrees $n-1$ and $m-1$, respectively. The product pf is a linear combination of the polynomials $x^i y^j f$, with $i+j = n-1$, and it will be equal to the linear combination qg of the polynomials $x^k y^\ell g$, with $k+\ell = m-1$. This implies that the $m+n$ polynomials of degree $m+n-1$,

mpluspolys

$$(1.5.3) \quad x^{n-1}f, x^{n-2}yf, \dots, y^{n-1}f; x^{m-1}g, x^{m-2}yg, \dots, y^{m-1}g$$

are dependent. For example, when $m = 3$ and $n = 2$, the polynomials $xf, yf; x^2g, xyg, y^2g$ will be dependent. Conversely, if the polynomials (1.5.3) are dependent, there will be an equation of the form $pf = qg$, with p of degree $n-1$ and q of degree $m-1$. At least one zero of g must be a zero of f , so f and g have a common zero.

We form a square matrix \mathcal{R} , the *resultant matrix*, whose columns are indexed by the $m+n$ monomials in x, y of degree $m+n-1$, and whose rows list the coefficients of those monomials in the polynomials (1.5.3). The matrix is illustrated below for the cases $m, n = 1, 2$ and $m, n = 3, 2$, with dots representing entries equal to zero:

resmatrix

$$(1.5.4) \quad \mathcal{R} = \begin{pmatrix} a_0 & a_1 & \cdot \\ \cdot & a_0 & a_1 \\ b_0 & b_1 & b_2 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \cdot \\ \cdot & a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & \cdot & \cdot \\ \cdot & b_0 & b_1 & b_2 & \cdot \\ \cdot & \cdot & b_0 & b_1 & b_2 \end{pmatrix}$$

This resultant matrix is defined when the coefficients are in any ring, and by definition, its determinant is the *resultant*:

resequalsdet

$$(1.5.5) \quad \text{Res}(f, g) = \det \mathcal{R}$$

The *resultant* $\text{Res}(F, G)$ of the one-variable polynomials (1.5.1) is the determinant of the matrix \mathcal{R} when $a_0 = b_0 = 1$.

homogresult

1.5.6. Corollary. *Let f and g be homogeneous polynomials in two variables, or monic polynomials in one variable, with coefficients in a field. The resultant $\text{Res}(f, g)$ is zero if and only if f and g have a common factor.* \square

When defining the degree of a polynomial, one may assign an integer called a *weight* to each variable. If one assigns weight w_i to the variable x_i , the monomial $x_1^{e_1} \cdots x_n^{e_n}$ gets a *weighted degree*, which is $w_1 e_1 + \cdots + w_n e_n$. For example, it is natural to assign weight k to the coefficient a_k of the polynomial $f(x) = x^n - a_1 x^{n-1} + a_2 x^{n-2} - \cdots \pm a_n$. The reason is that, if f factors into linear factors, say $f(x) = (x - \alpha_1) \cdots (x - \alpha_n)$, a_k will be the k th elementary symmetric function in $\alpha_1, \dots, \alpha_n$, and when written as a polynomial in α , the degree of a_k will be k .

degresult

1.5.7. Lemma. *Let f and g be homogeneous polynomials (1.5.2) with indeterminate coefficients. Assigning weight i to the coefficients a_i and b_i , the resultant $\text{Res}(f, g)$ is a weighted homogeneous polynomial of degree mn in the variables $\{a_i, b_j\}$.* \square

The next proposition gives a formula for the resultant of two monic polynomials in terms of their roots.

resroots

1.5.8. Proposition. *Let F and G be monic polynomials of degrees m and n , that are products of linear polynomials, say $F = \prod_i (x - \alpha_i)$ and $G = \prod_j (x - \beta_j)$. Then*

$$\text{Res}(F, G) = \prod_{i,j} (\alpha_i - \beta_j) = \prod_{i=1}^m G(\alpha_i) = \pm \prod_{i=1}^n F(\beta_i).$$

proof. The sign is $(-1)^{mn}$. It is easy to see that the three products are equal. We prove that the resultant $R = \text{Res}(F, G)$ is equal to the first product $\Pi = \prod_{i,j} (\alpha_i - \beta_j)$.

Let α_i and β_j be variables. The coefficient a_ν of x^ν in F is the elementary symmetric function of degree ν in the roots α_i , and the coefficient b_ν of x^ν in G is the elementary symmetric function of degree ν in the roots β_i . When we write those coefficients in terms of indeterminate α_i and β_j , the resultant R will be homogeneous. Its (unweighted) degree in α_i, β_j will be mn , the same as the degree of Π .

To show that $R = \Pi$, we choose i, j and divide the resultant by the polynomial $\alpha_i - \beta_j$, considered as a monic polynomial in α_i

$$R = (\alpha_i - \beta_j)q + r,$$

where r has degree zero in α_i . Since R vanishes when $\alpha_i = \beta_j$, so does the remainder r . Since r is independent of α_i , it isn't changed when we set $\alpha_i = \beta_j$, and therefore it is zero. This being true for all i and j , R is divisible by Π , and since these two polynomials have the same degree, $R = c\Pi$ for some scalar c . To evaluate c , one may compute R and Π for some particular polynomials, such as for $F = x^m$ and $G = x^n - 1$. \square

leadco-
effnotzero

1.5.9. Corollary. *Let f and g be homogeneous polynomials of the form (1.5.2), with complex coefficients. Then $\text{Res}(f, g) = 0$ if and only if either $f(x, 1)$ and $g(x, 1)$ have a common root, or a_0 and b_0 are both zero.* \square

When a_0 and b_0 are zero, the point $(1, 0)$ of \mathbb{P}^1 will be a zero of f and of g . In this case, f and g have a common zero at infinity.

discrimsect

(1.5.10) the discriminant

The *discriminant* $\text{Discr}^m(F)$ of a polynomial $F = a_0 x^m + a_1 x^{m-1} + \cdots + a_m$ is the resultant of F and its derivative F' :

discrdef

$$(1.5.11) \quad \text{Discr}^m(F) = \text{Res}(F, F')$$

The superscript m is there to avoid ambiguity, and we usually omit it. It indicates that the computation is made using the formula for the resultant $\text{Res}(F, F')$ of a polynomial F of degree m . The definition makes

sense when the leading coefficient a_0 is zero, but then $\text{Discr}^m(F) = 0$. It might be more useful to compute $\text{Discr}^{m-1}(F)$ in that case.

When the coefficients a_i are complex numbers, the discriminant is zero if and only if either F has a double root, which happens when F and F' have a common factor, or else F has degree less than m (see (1.5.9)).

The discriminant of the quadratic polynomial $F(x) = ax^2 + bx + c$ is

discrquadr (1.5.12)
$$\det \begin{pmatrix} a & b & c \\ 2a & b & \cdot \\ \cdot & 2a & b \end{pmatrix} = -a(b^2 - 4ac).$$

It is zero if and only if F has a double root, or has degree less than 2.

discrnotzero **1.5.13. Proposition.** *The discriminant of an irreducible polynomial F with coefficients in a field K of characteristic zero isn't zero. Therefore, an irreducible polynomial over a field of characteristic zero has no multiple root.*

proof. When F is irreducible, it cannot have a factor in common with the lower degree polynomial F' . This would be false if K had characteristic p . In characteristic p , the derivative F' may be the zero polynomial. \square

discrformulas **1.5.14. Proposition.** *Let $F = (x - \alpha_1) \cdots (x - \alpha_n)$ be a monic polynomial of degree d that is a product of linear factors. Then*

$$\text{Discr}(F) = \prod_j F'(\alpha_j) = \pm \prod_{i < j} (\alpha_i - \alpha_j)^2.$$

proof. The first equality follows from Proposition 1.5.8. To prove the second one, we use the product formula for the derivative to write

derprod (1.5.15)
$$F'(x) = \sum_j (x - \alpha_1) \cdots (x - \alpha_j)^\wedge \cdots (x - \alpha_d),$$

where \wedge indicates that the term $(x - \alpha_j)$ is omitted. For example, the derivative of $(x - \alpha)(x - \beta)(x - \gamma)$ is

$$(x - \beta)(x - \gamma) + (x - \alpha)(x - \gamma) + (x - \alpha)(x - \beta)$$

Substituting $x = \alpha_i$ into the sum (1.5.15), all terms except the one in which $i = j$ become zero, and

$$F'(\alpha_i) = (\alpha_i - \alpha_1) \cdots (\alpha_i - \alpha_i)^\wedge \cdots (\alpha_i - \alpha_d) = \prod_{j|j \neq i} (\alpha_i - \alpha_j)$$

\square

discrprop **1.5.16. Proposition.** $\text{Discr}(FG) = \pm \text{Discr}(F) \text{Discr}(G) (\text{Res}(F, G))^2.$

proof. This proposition follows from Propositions 1.5.8 and 1.5.14 for polynomials with complex coefficients. It is true for polynomials with coefficients in any ring because it is an identity. \square

hensel (1.5.17) **Hensel's Lemma**

The resultant matrix (1.5.4) arises in a second context that we explain here. Suppose given a product $P = FG$ of two polynomials, say

multiply-polys (1.5.18)
$$c_0x^{m+n} + c_1x^{m+n-1} + \cdots + c_{m+n} = (a_0x^m + a_1x^{m-1} + \cdots + a_m)(b_0x^n + b_1x^{n-1} + \cdots + b_n)$$

We call the equations among the coefficients that are implied by the equation $P = FG$ the *product equations*. When $m = 3$ and $n = 2$, the product equations are

prodeqns **1.5.19.**

$$\begin{aligned} c_0 &= a_0 b_0 \\ c_1 &= a_1 b_0 + a_0 b_1 \\ c_2 &= a_2 b_0 + a_1 b_1 + a_0 b_2 \\ c_3 &= a_3 b_0 + a_2 b_1 + a_1 b_2 \\ c_4 &= a_3 b_1 + a_2 b_2 \\ c_5 &= a_3 b_2 \end{aligned}$$

Let J denote the Jacobian matrix of partial derivatives of c_1, \dots, c_{m+n} with respect to the variables b_1, \dots, b_n and a_1, \dots, a_m , holding a_0, b_0 and c_0 constant. When $m, n = 3, 2$,

prodjacob (1.5.20)
$$J = \frac{\partial(c_i)}{\partial(b_j, a_k)} = \begin{pmatrix} a_0 & \cdot & b_0 & \cdot & \cdot \\ a_1 & a_0 & b_1 & b_0 & \cdot \\ a_2 & a_1 & b_2 & b_1 & b_0 \\ a_3 & a_2 & \cdot & b_2 & b_1 \\ \cdot & a_3 & \cdot & \cdot & b_2 \end{pmatrix}$$

This Jacobian matrix is the transpose of the resultant matrix \mathcal{R} (1.5.4).

jacobian-notzero **1.5.21. Corollary.** *With F and G as above, the Jacobian matrix is singular if and only if F and G have a common root, or $a_0 = b_0 = 0$.*

This follows from Corollary 1.5.9. □

We apply Corollary 1.5.21 to polynomials with analytic coefficients. Let

polyforhensel (1.5.22)
$$P(t, x) = c_0(t)x^d + c_1(t)x^{d-1} + \dots + c_d(t)$$

be a polynomial in x whose coefficients c_i are analytic functions, defined for small values of t , and let $\bar{P} = P(0, x) = \bar{c}_0 x^d + \bar{c}_1 x^{d-1} + \dots + \bar{c}_d$ be the evaluation of P at $t = 0$, so that $\bar{c}_i = c_i(0)$. Suppose given a factorization $\bar{P} = \bar{F}\bar{G}$, where $\bar{F} = x^m + \bar{a}_1 x^{m-1} + \dots + \bar{a}_m$ is a monic polynomial and $\bar{G} = \bar{b}_0 x^n + \bar{b}_1 x^{n-1} + \dots + \bar{b}_n$ is another polynomial, both with complex coefficients. We ask whether this factorization of \bar{P} is induced by a factorization of P . Are there polynomials $F(t, x) = x^m + a_1 x^{m-1} + \dots + a_m$ and $G(t, x) = b_0 x^n + b_1 x^{n-1} + \dots + b_n$, with F monic, whose coefficients a_i and b_i are analytic functions defined for small t , such that $\bar{F} = F(0, x)$, $\bar{G} = G(0, x)$, and that $P = FG$?

figure

hensellemma **1.5.23. Hensel's Lemma.** *With notation as above, suppose that \bar{F} and \bar{G} have no common roots. Then P factors, as above.*

proof. Since F is supposed to be monic, we set $a_0 = 1$. The first product equation tells us that $b_0 = c_0$. Corollary 1.5.21 tells us that the Jacobian matrix for the remaining product equations is nonsingular at $t = 0$, so according to the Implicit Function Theorem, the product equations have a unique solution in analytic functions $a_i(t), b_j(t)$ for small t . □

Note that P is not assumed to be monic. If $\bar{c}_0 = 0$, the degree of \bar{P} will be less than the degree of P . In that case, \bar{G} will have lower degree than G .

coverline **1.6 Plane Curves as Coverings of the Projective Line**

Let π denote the *projection* $\mathbb{P}^2 \rightarrow \mathbb{P}^1$ that drops the last coordinate, sending a point (x, y, z) to (x, y) . This projection is defined at all points of \mathbb{P}^2 except at the point $q = (0, 0, 1)$. The fibre of π over a point $\tilde{p} = (x_0, y_0)$ of \mathbb{P}^1 is the line $L_{\tilde{p}}$ through the points $(x_0, y_0, 0)$ and q , with the point q omitted, the set of points (x_0, y_0, z) .

figure

firroverK

1.6.1. Lemma. Let $f(x, y, z)$ be an irreducible polynomial in $\mathbb{C}[x, y, z]$, and let K denote the rational function field $\mathbb{C}(x, y)$.

(i) f is an irreducible element of $K[z]$.

(ii) The discriminant $\text{Discr}_z(f)$ of f with respect to the variable z is a nonzero polynomial in x, y .

proof. (i) Say that a polynomial $f(x, y, z)$ factors in $K[z]$, $f = GH$, where G and H are nonconstant polynomials in z with coefficients in K . The coefficients of F and G have denominators that are polynomials in x, y . When we clear those denominators, we obtain an equation in $\mathbb{C}[x, y, z]$ of the form $df = gh$, where d is a polynomial in x, y and where g and h are polynomials in x, y, z that have positive degree in z . Since $\mathbb{C}[x, y, z]$ has unique factorization, f must be reducible.

(ii) This is Proposition 1.5.13. □

We go back to the projection $\mathbb{P}^2 \xrightarrow{\pi} \mathbb{P}^1$. Suppose that a curve $C : \{f = 0\}$ in \mathbb{P}^2 doesn't pass through q . We write f as a polynomial in z :

polyinztwo

$$(1.6.2) \quad f = a_0 z^d + a_1 z^{d-1} + \cdots + a_d,$$

with a_i of degree i in x, y . Since C doesn't contain q , the coefficient a_0 of z^d will be a nonzero constant which we normalize to 1.

The projection π will be defined everywhere on C . The fibre of C over a point $\tilde{p} = (x_0, y_0)$ of \mathbb{P}^1 is the intersection of C with the line $L_{\tilde{p}}$ described above. It consists of the points (x_0, y_0, α) such that α is a root of the one-variable polynomial

effp

$$(1.6.3) \quad f_{\tilde{p}}(z) = f(x_0, y_0, z).$$

Since f is a monic polynomial of degree d , $f_{\tilde{p}}(z)$ is a monic polynomial of degree d in z , for every point \tilde{p} . Its discriminant $\text{Discr}_z(f_{\tilde{p}})$ is obtained by evaluating $\text{Discr}_z(f)$ at \tilde{p} , and it will be nonzero for all but finitely many points \tilde{p} of \mathbb{P}^1 . If $\text{Discr}_z(f_{\tilde{p}})$ is nonzero, $f_{\tilde{p}}$ has d roots. If $\text{Discr}_z(f_{\tilde{p}})$ is zero, then $f_{\tilde{p}}$ has a multiple root. In that case, the line $L_{\tilde{p}}$ will be tangent to C or will contain a singular point of C . Because all but finitely many fibres consist of d points, C is called a *d-sheeted branched covering* of \mathbb{P}^1 . The *branch points* are the points \tilde{p} at which the discriminant $\text{Discr}_z(f)$ is zero, those such that the fibre over \tilde{p} has fewer than d points.

curveinfinite

1.6.4. Corollary. (i) A plane curve contains infinitely many points, and finitely many singular points.

(ii) With its classical topology, a plane curve contains no isolated points.

proof. (ii) A point p of a topological space X is *isolated* if both $\{p\}$ and its complement $X - \{p\}$ are closed in X . Another way to say this is that p is an isolated point if X doesn't contain points p' distinct from p , but arbitrarily close to p .

Let p be a point of C . We may assume that C is the locus of zeros of the polynomial (1.6.2) with a_0 normalized to 1, and that $p = (1, 0, 0)$. Let $\tilde{y} = (1, y)$ be a point of \mathbb{P}^1 near to $\tilde{p} = (1, 0)$, and let $f_{\tilde{y}}(z) = f(1, y, z)$. Since p is a point of C , $a_d(1, 0) = 0$. So the constant coefficient $a_d(1, y)$ of $f_{\tilde{y}}$ will tend to zero with y . Because the constant coefficient is the product of the roots, at least one root α' of $f_{\tilde{y}}$ will approach zero. This gives us points $(1, y, \alpha')$ of C arbitrarily close to p . □

function-
iszero

1.6.5. Corollary. Let U be the complement of a finite set of points in a plane curve C . A continuous function g on C that is zero at every point of U is identically zero. This is true in the Zariski topology as well as in the classical topology. □

The next lemma determines the order of vanishing of the discriminant at the images \tilde{p} of the smooth points p whose tangent lines contain q .

discrimvan-
ishing

1.6.6. Lemma. Let $C : \{f = 0\}$ be a smooth plane curve, and let $C \xrightarrow{\pi} \mathbb{P}^1$ be the projection with center $q = (0, 0, 1)$. Let \tilde{p} be a point of \mathbb{P}^1 . Suppose that the line $L_{\tilde{p}}$ is a tangent line to C at a single point p , and that the order of contact of \tilde{p} and C at q is 2. Then the discriminant $\text{Discr}_z(f)$ has a simple zero at \tilde{p} .

proof. We set $z = 1$, to work in the standard affine open set U with coordinates x, y . In affine coordinates, the projection is the map $(x, y) \rightarrow y$. We adjust coordinates so that p is the origin $(0, 0)$ in U and its image is the

point $\tilde{p} : y = 0$. Let L be the line $\{y = 0\}$ in U , and let $f(x, y)$ denote the defining polynomial of the curve C , restricted to U .

The polynomial $\bar{f}(x) = f(x, 0)$ will have a double zero at $x = 0$. We will have $\bar{f} = x^2\bar{h}(x)$, with $\bar{h}(0) \neq 0$, and the coefficient of x^2 in f will be nonzero. Since x^2 and \bar{h} have no root in common, we may apply Hensel's Lemma 1.5.23 to write $f(x, y) = g(x, y)h(x, y)$, where g and h are polynomials in x whose coefficients are analytic functions of y , defined for small y , g is monic, $g(x, 0) = x^2$, and $h(x, 0) = \bar{h}$. According to Proposition 1.5.16,

$$\text{Discr}_x(f) = \text{Discr}_x(g) \text{Discr}_x(h) (\text{Res}_x(g, h))^2.$$

The contribution of p to the vanishing of $\text{Discr}_x(f)$ is the order of vanishing of $\text{Discr}_x(g)$. We can replace f by g , and doing so reduces us to the case that f is a quadratic polynomial in x , say

$$f(x, y) = a(y)x^2 + b(y)x + c(y),$$

where a, b, c are analytic functions of y and $f(x, 0) = x^2$. We write these functions as series in y : $a(y) = a_0 + a_1y + a_2y^2 \cdots$, etc. Since $f(x, 0) = x^2$, the constant terms are $a_0 = 1$, and $b_0 = c_0 = 0$. Since C is smooth, $c_1 \neq 0$. Then $f = c_1y + O((x, y)^2)$, and the discriminant $D = b^2 - 4ac = -4c_1y + O(y^2)$ has a zero of order one.

bezoutthm **1.7 Bézout's Theorem**

Bézout's Theorem counts intersections of plane curves. We state it here in a form that is ambiguous because it contains an as yet undefined term "multiplicity".

bezoutone **1.7.1. Bézout's Theorem.** Let C and D be distinct curves of degrees m and n , respectively. When counted with an appropriate multiplicity, the number of points of intersection of C and D is equal to mn . Moreover, the multiplicity at a point p is 1 if C and D intersect transversally at p .

Two curves intersect *transversally* at a point p if they are smooth at p and if their tangent lines there are distinct.

The proof of Bézout's Theorem requires some algebra that we would rather defer. It will be given later (??). It is possible to determine the intersections by counting the zeros of the resultant with respect to one of the variables. To do this, one chooses coordinates x, y, z so that neither f nor g is zero at the point $(0, 0, 1)$, and one writes f and g as polynomials in z with coefficients in $\mathbb{C}[x, y]$. The resultant with respect to the variable z will be a homogeneous polynomial $R(x, y)$ in x, y , of degree mn . It will have mn zeros in $\mathbb{P}_{x,y}^1$, counted with multiplicity (see Section 1.3). If $\tilde{p} = (x_0, y_0)$ is a zero of R , the one-variable polynomials $f(x_0, y_0, z)$ and $g(x_0, y_0, z)$ have a common root $z = z_0$, and then $p = (x_0, y_0, z_0)$ will be a point of $C \cap D$. Unfortunately, it isn't obvious that the multiplicity of the zero of R at \tilde{p} is the (as yet undefined) multiplicity of intersection of C and D at p , or even that it is independent of the choice of coordinates. However, we can prove the next proposition using this approach.

nocommon-factor **1.7.2. Proposition.** Two projective plane curves have at least one point of intersection, and distinct plane curves of degrees m and n have at most mn intersections.

It isn't obvious that two curves in the projective plane must intersect. Curves in the affine plane, such as parallel lines, may have no intersection. In that case, their closures in the projective plane meet on the line at infinity.

resnotzero **1.7.3. Lemma.** Let f and g be irreducible homogeneous polynomials in x, y, z with complex coefficients, and suppose that the point $(0, 0, 1)$ isn't a zero of f or g . If the resultant $R = \text{Res}_z(f, g)$ with respect to z is identically zero, then f and g have a common factor.

proof. Let the degree of f and g be m and n respectively, and let K denote the field of rational functions $\mathbb{C}(x, y)$. If the resultant R is zero, then f and g have a common factor in $K[z]$ (Corollary 1.5.6). So there are polynomials p and q in $K[z]$, of degrees at most $n - 1$ and $m - 1$ in z respectively, such that $pf = qg$. We may clear denominators, so we may assume that the coefficients of p and q are in $\mathbb{C}[x, y]$. Then $pf = qg$ is an equation in $\mathbb{C}[x, y, z]$. Since p has degree $n - 1$ in z , it isn't divisible by g . Since $\mathbb{C}[x, y, z]$ is a unique factorization domain, f and g have a common factor. \square

proof of Proposition 1.7.2. Let C and D be distinct curves, the zero sets of the irreducible homogeneous polynomials f and g . Then f and g have finitely many common zeros. Their common zeros lie over the zeros of the resultant R with respect to z , which is a nonzero homogeneous polynomial of degree mn in x, y (Lemma 1.5.7). So $C \cap D$ is a finite set, and it is nonempty. This being so, we project from a point q that doesn't lie on any of the finitely many lines through pairs of intersection points. Then a line through q passes through at most one intersection, and the zeros of the resultant that correspond to the intersection points will be distinct. Since the resultant has degree mn , there can be at most mn such zeros, and at most mn intersections. \square

1.7.4. Corollary. (i) *If the divisor X defined by a homogeneous polynomial $f(x, y, z)$ is smooth, then f is irreducible, and therefore X is a plane curve.*

(ii) *There exist irreducible homogeneous polynomials in three variables, of arbitrary degree.*

proof. (i) Suppose that $f = gh$, and let p be a point of intersection of the loci $\{g = 0\}$ and $\{h = 0\}$. The previous proposition shows that such a point exists. The partial derivatives f_i are zero at p , so p is a singular point of X .

(ii) The Fermat polynomial $x^d + y^d + z^d$ is irreducible because its locus of zeros is smooth. \square

The next corollary is a special case of the Strong Nullstellensatz that will be proved in the next chapter.

1.7.5. Corollary. *Let S be an infinite set of points of \mathbb{P}^2 , and let f be an irreducible homogeneous polynomial that vanishes on S . If another homogeneous polynomial g vanishes on S , then f divides g . Therefore, if an irreducible polynomial vanishes on S , that polynomial is unique up to scalar factor.*

proof. If the irreducible polynomial f does not divide g , then f and g have no common factor, and therefore they have finitely many common zeros. \square

1.7.6. Proposition. *Bézout's Theorem is true when one of the curves is a line.*

proof. Let C denote the locus $\{f = 0\}$, where f is an irreducible homogeneous polynomial of degree m , and let D be a line. We choose distinct points p, q of D , so that D is the set $\{z_0p + z_1q\}$, and we identify D with the projective line by $z_0p + z_1q \leftrightarrow (z_0, z_1)$. Restricting f to D gives us a homogeneous polynomial $\bar{f}(z_0, z_1) = f(z_0p + z_1q)$ of degree m in the variables z_0, z_1 , and $\bar{f}(z_0, z_1) = 0$ if and only if the point $z_0p + z_1q$ lies on C . The zeros of \bar{f} are the intersections of C with D .

According to Lemma 1.3.4, $\bar{f}(z_0, z_1)$ factors into m linear factors. A point (a, b) of \mathbb{P}^1 is a zero of $\bar{f}(z_0, z_1)$ if and only if $az_1 - bz_0$ is a linear factor, and the multiplicity of this zero is the exponent of the largest power $(az_1 - bz_0)^k$ that divides \bar{f} . When counted with this multiplicity, the number of zeros is the degree m of \bar{f} , which is also the degree of C . Since D has degree 1, this is Bézout's Theorem. \square

1.7.7. Corollary. *A smooth plane curve of degree $d > 2$ has at least one flex point, and the number of flex points is at most $3d(d-2)$.*

proof, assuming Bézout's Theorem Let $C : \{f = 0\}$ be a smooth curve of degree d . The entries of the 3×3 Hessian matrix H are the second partial derivatives $\frac{\partial^2 f}{\partial x_i \partial x_j}$. They are homogeneous polynomials of degree $d-2$, so the Hessian determinant is homogeneous, of degree $3(d-2)$. The intersections of a curve C with the Hessian divisor $D : \{\det H = 0\}$ are either flex points or singular points. Since C is smooth, they are flex points. Proposition 1.7.2 tells us that there are at most $3d(d-2)$ intersections. \square

Thus the numbers of flex points of smooth curves of degrees 2, 3, 4, 5, ... are at most 0, 9, 24, 45, ..., respectively. Bézout's Theorem also asserts that the number of flex points is precisely $3d(d-2)$ if the intersections of C with its Hessian divisor D are transversal, and therefore have multiplicity 1. The next proposition explains the meaning of transversality.

1.7.8. Proposition. *A curve $C : \{f = 0\}$ intersects the Hessian divisor D transversally at a flex point p if and only if p is an ordinary flex, meaning that C and its tangent line have a contact of order precisely 3 at p .*

proof. I don't know how to show this except by making the computation. The order of contact of C with its tangent line ℓ at a point p is the order of vanishing of the one-variable polynomial obtained by restricting f to ℓ . The Hessian divisor D will be transversal to C at p if and only if it is transversal to ℓ , i.e., if and only if the order of vanishing of the Hessian determinant, restricted to ℓ , is 1.

We adjust coordinates x, y, z so that the flex point is $p = (0, 0, 1)$ and the tangent line at p is the line $\ell : \{y = 0\}$. We write the polynomial f of degree d as

$$(1.7.9) \quad f(x, y, z) = \sum_{i+j+k=d} a_{ij} x^i y^j z^k,$$

The restriction of f to ℓ is the polynomial $f(x, 0, z) = \sum a_{i0} x^i z^k$. Since p is a flex point with tangent line ℓ , $a_{00} = a_{10} = a_{20} = 0$, and p is an ordinary flex if and only if the coefficient a_{30} is nonzero.

We set $y = 0$ and $z = 1$ in the second partial derivatives and we eliminate terms of degree greater than one in the remaining variable x . The results are

$$\begin{aligned} f_{xx}(x, 0, 1) &= 6a_{30}x \\ f_{xz}(x, 0, 1) &= 0 \\ f_{yz}(x, 0, 1) &= (d-1)a_{01}(d-2)a_{11}x \\ f_{zz}(x, 0, 1) &= 0 \end{aligned}$$

Writing $\beta = (d-1)a_{01} + (d-2)a_{11}x$, the resultant matrix has the form

$$(1.7.10) \quad H_p = \begin{pmatrix} 6a_{30}x & * & 0 \\ * & * & \beta \\ 0 & \beta & 0 \end{pmatrix} + O(x^2)$$

where $*$ are undetermined entries. Its determinant has the form $6(d-1)^2 a_{30} a_{01}^2 x + O(x^2)$. It has a zero of order 1 at $x = 0$ if and only if a_{30} and a_{01} aren't zero. Since C is smooth at p and $a_{01} = 0$, the coefficient a_{01} isn't zero. Thus a smooth point p is an ordinary flex if and only if $a_{30} \neq 0$. \square

1.7.11. Corollary. *A smooth cubic curve C contains exactly 9 flex points.*

proof, assuming Bézout's Theorem. Let f be the irreducible cubic polynomial whose zero locus is the cubic C . The degree of the Hessian divisor D is also 3, so Bézout predicts at most 9 intersections of D with C . To derive the corollary, we show that C intersects D transversally. According to Proposition 1.7.8, a nontransversal intersection would correspond to a point at which the curve and its tangent line have a contact of order greater than 3. This is impossible when the curve is a cubic. \square

genus 1.8 Genus

In this section, we describe the topological structure of smooth plane curves in the classical topology. So we work in the classical topology here.

1.8.1. Theorem. *The smooth plane curves of degree d in \mathbb{P}^2 are homeomorphic manifolds of dimension two. They are orientable, compact, and connected.*

connectedness: Unfortunately, this is a subtle point that mixes topology and algebra, and I don't know a proof that fits into our discussion. It will be proved later (see 8.3.7).

One can show that the Fermat curve $x^d + y^d + z^d = 0$ is connected by studying the projection to \mathbb{P}^1 from the point $(0, 0, 1)$. I'm proposing this as an exercise. Then because the set of points of Z corresponding to smooth curves of degree d is path connected, one can show that every such curve is connected by showing that if a family C_t of smooth projective curves of degree d is parametrized by t in an interval of the real line, the curves in the family are homeomorphic. This can be done by constructing a *gradient flow*. This approach has two drawbacks: It leads us far afield, and it applies only to plane curves. If you are interested in following this up, read about gradient flows.

Orientability: A two-dimensional manifold is orientable if one can choose one of its two sides in a continuous, consistent way. A smooth curve C is orientable because its tangent space at a point is a one-dimensional complex vector space, the affine line with the equation (1.4.6). Multiplication by i orients the tangent spaces by defining the counterclockwise rotation. Then the "right-hand rule" tells us which side of C is "up".

compactness: A curve is compact because it is a closed subset of the compact space \mathbb{P}^2 .

The *Euler characteristic* e of a compact, connected, orientable two-dimensional manifold X is the alternating sum $b^0 - b^1 + b^2$ of the Betti numbers of the manifold. It depends only on the topological structure of the manifold, and it can be computed in terms of a *triangulation*, a subdivision of the manifold into topological triangles, called *faces*, by the formula

vef (1.8.2)
$$e = |\text{vertices}| - |\text{edges}| + |\text{faces}|$$

For example, a tetrahedron is homeomorphic to a sphere. It has four vertices, six edges, and four faces, so its Euler characteristic is $4 - 6 + 4 = 2$. Any other topological triangulation of a sphere, such as the one given by the icosahedron, yields the same Euler characteristic.

Every compact, *connected*, orientable two-dimensional manifold X is homeomorphic to a sphere with a finite number of “handles”. Its *genus* is the number of handles. For instance, a torus has one handle. Its genus is one. The projective line \mathbb{P}^1 , a sphere, has genus zero.

The Euler characteristic and the genus are related by the formula

genuseuler (1.8.3)
$$e = 2 - 2g.$$

The Euler characteristic of a torus is zero, and the Euler characteristic of \mathbb{P}^1 is two.

To compute the the Euler characteristic of a smooth curve C of degree d , we analyze a generic projection $C \xrightarrow{\pi} \mathbb{P}^1$.

figure

We choose coordinates x, y, z in \mathbb{P}^2 and project from the point $q = (1, 0, 0)$. We write the defining equation of C as a polynomial in x , say

$$f = a_0x^d + a_1x^{d-1} + \cdots + a_d$$

The discriminant $\text{Discr}_x(f)$ with respect to x is a polynomial of degree $d(d-1)$ in the variables y, z .

The covering $C \xrightarrow{\pi} \mathbb{P}^1$ will be branched at a point p of C if the tangent line at p contains q . When the coordinates are chosen generically, the tangent lines through q will be ordinary tangents. They will not be tangents at flex points, and they will be tangent to C at just one point. They will not be “bitangents”. It is intuitively plausible that a plane curve can have only finitely many bitangents. We will prove this later (1.12.14).

generic **1.8.4. Note.** In algebraic geometry, the phrases *general position* and *generic* indicate some object (a projection here) that has no special ‘bad’ properties. Typically, the object will be parametrized somehow, and the word generic indicates that the parameter representing the particular object avoids a proper subset that may or may not be described explicitly. In our case, the requirement is that q shall not be contained in any of the finitely many bitangents and flex tangents. □

If \tilde{p} is the image of such a tangent line, Lemma 1.6.6 tells us that the discriminant $\text{Discr}_x(f)$ will have a simple zero at the image of such a tangent line. So there will be $d(d-1)$ points \tilde{p} in \mathbb{P}^1 over which the fibre of the map has order $d-1$. They are the branch points of the covering.

We triangulate \mathbb{P}^1 in such a way that the branch points \tilde{p} are among the vertices, and we use the inverse images to triangulate C . Then C will have d faces, d edges, and d vertices over each face, edge, and vertex of \mathbb{P}^1 , respectively, except that there will be only $d-1$ vertices over each of the $d(d-1)$ branch points. Therefore the Euler characteristic of C is

$$e(C) = de(\mathbb{P}^1) - d(d-1) = 3d - d^2.$$

This is the Euler characteristic of any smooth curve of degree d , so we denote it by e_d :

equatione (1.8.5)
$$e_d = 3d - d^2.$$

The genus g_d of a smooth curve of degree d is therefore

equationg (1.8.6)
$$g_d = \frac{1}{2}(d-1)(d-2) = \binom{d-1}{2}.$$

Smooth curves of degrees 1, 2, 3, 4, 5 and 6 have genus 0, 0, 1, 3, 6 and 10, respectively. A smooth plane curve cannot have genus 2.

1.9 Projective Space is Proper

An important property of the projective plane is that, with its classical topology, it is a *compact space*. A topological space is compact if it has these properties:

It is a *Hausdorff space*: Distinct points p, q of X have disjoint open neighborhoods.

It is *quasicompact*: If X is covered by a family $\{U_i\}$ of open sets, then a finite subfamily covers X .

The *Heine-Borel Theorem* asserts that a subset of \mathbb{R}^n is compact if and only if it is closed and bounded.

We omit the verification that, with its classical topology, \mathbb{P}^2 is a Hausdorff space. To prove that \mathbb{P}^2 is compact, we note that the five-dimensional sphere \mathbb{S} of unit length vectors in \mathbb{C}^3 is bounded, and because it is the zero locus of the equation $\bar{x}_0x_0 + \cdots + \bar{x}_n x_n = 1$, it is closed. So \mathbb{S} is compact. The map $\mathbb{S} \rightarrow \mathbb{P}^2$ that sends a vector (x_0, x_1, x_2) to the point of the projective plane with that coordinate vector is continuous and surjective. The next lemma tells us that \mathbb{P}^2 is compact. Similarly, \mathbb{P}^n is compact for every n .

imagecompact

1.9.1. Lemma. *Let $X \xrightarrow{f} Y$ be a continuous map. Suppose that X is a compact space and that Y is a Hausdorff space. Then the image $f(X)$ is a closed subset of Y , and with the topology induced from Y , the image is compact.* \square

We have seen that, in the Zariski topology, a projective space isn't Hausdorff. It is quasicompact, but not compact. However, it has a closely related property: It is *proper*. To explain this property, we need to define Zariski closed subsets of a product of projective spaces.

Let $x = x_0, \dots, x_m$ and $y = y_0, \dots, y_n$ denote the coordinates in two projective spaces \mathbb{P}^m and \mathbb{P}^n , respectively. A polynomial $f(x, y)$ is *bihomogeneous* if it is homogeneous in x and also in y . For example, the polynomial $x_0^2 y_0 + x_0 x_1 y_1$ is bihomogeneous, of degree 2 in x and degree 1 in y . If f is bihomogeneous, of degree r in x and degree s in y , then $f(\lambda x, \mu y) = \lambda^r \mu^s f(x, y)$. It makes sense to say that f does or is not zero at a point of $\mathbb{P}^m \times \mathbb{P}^n$.

defclnprod

1.9.2. Definition A subset V of the product $\mathbb{P}^m \times \mathbb{P}^n$ is *Zariski closed* if it is the set of common zeros of a family $f = f_1, \dots, f_k$ of bihomogeneous polynomials in x, y .

An example: Let \mathbb{P} denote the projective plane \mathbb{P} with coordinates x , and let u be coordinates in the dual plane \mathbb{P}^* . The locus of zeros of the equation $u_0 x_0 + u_1 x_1 + u_2 x_2 = 0$ in $\mathbb{P} \times \mathbb{P}^*$ is the closed set whose points are pairs (p, ℓ^*) such that p is a point of the line ℓ .

We discuss Zariski closed sets of products more fully in Section 3.1.

The next theorem will be proved in Chapter 5, but we make use of it here.

pspace-properone

1.9.3. Theorem: projective space is proper. *Let V be a Zariski closed subset of a product $\mathbb{P}^m \times \mathbb{P}^n$. The image of V in \mathbb{P}^m via the projection $\mathbb{P}^m \times \mathbb{P}^n \rightarrow \mathbb{P}^m$ is a Zariski closed subset of \mathbb{P}^m .*

variety-ofcurves

(1.9.4) the variety of curves of degree d

We assemble the plane curves of a given degree d into a variety. This provides an example of a closed subset of a product, and it shows how Theorem 1.9.3 can be used.

The number of distinct monomials $x_0^i x_1^j x_2^k$ of degree $d = i + j + k$ is the binomial coefficient $\binom{d+2}{2}$. We order the monomials arbitrarily, labeling them as m_0, \dots, m_r , $r = \binom{d+2}{2} - 1$. A homogeneous polynomial of degree d will be a combination

polydegd

$$(1.9.5) \quad f(x) = z_0 m_0 + \cdots + z_r m_r,$$

with complex coefficients z_i . So the homogeneous polynomials of degree d , taken up to scalar factors, are parametrized by a projective space with coordinates z_0, \dots, z_r . Let's denote that projective space by Z . Speaking loosely, we may say that Z is the space of homogeneous polynomials of degree d , or the space of curves of

degree d . A correct statement is that points of Z correspond bijectively to divisors of degree d in the projective plane (see Section 1.3.5).

The variable polynomial f exhibited in (1.9.5) may be written as $f(z, x)$. It is bihomogeneous, linear in z and of degree d in x .

The product space $Z \times \mathbb{P}^2$ represents pairs (f, p) where p is a point of \mathbb{P}^2 and $f(z, x)$ is a homogeneous polynomial of degree d , up to scalar factor, or pairs (D, P) , D being a divisor of degree d . So the locus $\Gamma: \{f(z, x) = 0\}$ in $Z \times \mathbb{P}^2$ is a (Zariski) closed set whose points are pairs (D, p) such that p is a point of the divisor D . The set Σ of pairs (D, p) such that p is a singular point of D is also closed. It is defined by the system of equations $f_0(z, x) = f_1(z, x) = f_2(z, x) = 0$, where f_i is the partial derivative $\frac{\partial f}{\partial x_i}$, as usual. The partial derivatives f_i are bihomogeneous, of degree 1 in z and degree $d - 1$ in x .

The next proposition isn't easy to prove directly, but it becomes easy to prove when one uses the fact that projective space is proper.

singclosed **1.9.6. Proposition** *The set S of singular divisors of degree d is a (Zariski) closed subset of the space Z of all curves of degree d .*

proof. The image via projection to Z of the subset Σ defined above is the set S of points of Z that correspond to singular divisors. heorem 1.9.3 tells us that S is closed. \square

nodes 1.10 Nodes and Cusps

We take a look here at the simplest singularities of curves, nodes and cusps.

Given a projective curve $C : \{f(x, y, z) = 0\}$, we choose coordinates so that the point to inspect is $p = (0, 0, 1)$, and we set $z = 1$. This gives us an affine curve C_0 , the zero set of the polynomial $\tilde{f}(x, y) = f(x, y, 1)$. We write

seriesf (1.10.1)
$$\tilde{f}(x, y) = f_0(x, y) + f_1(x, y) + f_2(x, y) + \dots,$$

where f_i is the homogeneous part of \tilde{f} of degree i , which is also the coefficient of z^i in the polynomial $f(x, y, z)$. If p is a point of C_0 , the constant term f_0 will be zero. Then the linear term f_1 will define the tangent direction to C_0 at p , unless it is zero too. If f_0 and f_1 are both zero, p will be a singular point.

To describe a singularity, one looks first at the homogeneous part of lowest degree. The smallest integer r such that $f_r \neq 0$ is called the *multiplicity* of the point p . (This is different from the multiplicity of zero that was defined in Section 1.3.)

A *double point* is a point of multiplicity 2, and a double point p is a *node* if the quadratic part

quadrat-icterm (1.10.2)
$$f_2 = ax^2 + bxy + cy^2$$

of f has distinct zeros in \mathbb{P}^1 . A node is the simplest singularity that a curve can have.

Suppose that p is a double point and that the discriminant $b^2 - 4ac$ of f_2 is zero. Then f_2 is a square. Provided that $a \neq 0$,

squarefactor (1.10.3)
$$(2ax + by)^2 = 4af_2.$$

The curve C_0 and the line $L : \{2ax + by = 0\}$ have an intersection of multiplicity at least 3 at p , and p is a *cuspid* if the intersection has multiplicity precisely 3, which will be true when the cubic term $f_3(x, y)$ isn't divisible by $2ax + by$. Cusps are the next simplest singularities, after nodes.

If p is a double point and if the discriminant is zero, one can adjust coordinates and scale f to make $f_2 = y^2$. Then p will be a cusp if the coefficient of the monomial x^3 in f_3 isn't zero. The *standard cusp* is the locus $y^2 = x^3$.

figure : ?trefoil knot?

The simplest example of a double point that isn't a node or cusp is a *tacnode*, a point at which two branches of a curve intersect with the same tangent direction.

blowupnode-
cusp

(1.10.4) blowing up nodes and cusps

A standard way to analyze a singular point of a curve is to see what happens to it when one blows up the plane. We use an affine blowing up here.

Let U be the x, y -plane, and let V be a second affine plane, with coordinates x, z . The *affine blowing up* of the origin $p = (0, 0)$ in U is the map $V \xrightarrow{\pi} U$ defined by $x = x$, and $y = xz$:

$$\pi(x, z) = (x, xz)$$

The fibre of π over the point $(0, y)$ is empty, when $y \neq 0$, and the fibre over (x, y) consists of the single point $(x, z) = (x, x^{-1}y)$ when $x \neq 0$.

The interesting fibre of π is the fibre over the origin $p = (0, 0)$, the line

$$Z : \{x = 0\}$$

Its points correspond to tangent directions at p in a way that will be explained below. The reason that π is referred to as a blowup is that the origin in U is replaced by a line in V . It would be more logical to say that π is a “blowing down”, but this isn’t customary.

Suppose that p is a point of multiplicity r an affine curve C_0 . The polynomial that defines C_0 will have the form

mult (1.10.5)
$$\tilde{f}(x, y) = f_r(x, y) + f_{r+1}(x, y) + \dots$$

We choose coordinates so that f_r isn’t divisible by x , and we substitute $y = xz$ into \tilde{f} . Then $f_r(x, xz) = x^r f_r(1, z)$, and all higher degree terms of \tilde{f} are divisible by x^{r+1} . Let $g(x, z) = x^{-r} \tilde{f}(x, xz)$. Then

blowup-
cusptwo (1.10.6)
$$g(x, z) = f_r(1, z) + xh(x, z)$$

for some polynomial h . The curve $\{g(x, z) = 0\}$ is the blowup C_1 of C_0 in V .

The points of C_1 that lie over p are the intersections of C_1 with the line Z . To compute those intersections, we set $x = 0$, obtaining the polynomial $g(0, z) = f_r(1, z)$. The intersections of Z with C_0 are the points $z = \alpha$, where α is a root of this polynomial. They correspond to the zeros of $f_r(x, y)$ in $\mathbb{P}_{x,y}^1$.

Suppose that p is a smooth point of C_0 , i.e., that $f_1 = ax + by$ is not zero, and $\tilde{f} = ax + by + \dots$. In this case, $g(0, z) = a + bz + \dots$, and C_1 will have just one intersection of with Z , at the point $z = a/b$. This scalar a/b is the slope of the tangent line $ax + by = 0$ to C_0 at p . So points of Z correspond to tangent directions, as asserted.

Next, suppose that p is a double point of an affine curve C_0 , so that

mult (1.10.7)
$$\tilde{f}(x, y) = f_2(x, y) + f_3(x, y) + \dots$$

Then

blowup-
cusptwo (1.10.8)
$$g(x, z) = f_2(1, z) + xh(x, z)$$

figure

The intersections of $C_1 : \{g = 0\}$ with the line Z determined by the two roots of this polynomial on Z , as aboe. They correspond to the zeros of $f_2(x, y)$ in $\mathbb{P}_{x,y}^1$.

blowupcusp

1.10.9. Proposition. *With assumptions and notation as above,*

(i) *The multiplicity of the point p is equal to the number of intersections, counted with multiplicity, of the blowup curve C_1 with the line Z .*

(ii) *A double point p is a node or a cusp if and only if the blowup C_1 is smooth at the points lying over p . If so, then*

- (a) *p is a node if $f_2(1, z)$ has two distinct roots, and*
- (b) *p is a cusp if $f_2(1, z)$ has a double root.*

proof. Because p is a double point, $C_1 \cap Z$ consists of two points of multiplicity one or one point of multiplicity two. Moreover, $C_1 \cap Z$ consists of two points of multiplicity 1 if and only if $f_2(x, y)$ has two zeros in \mathbb{P}^1 , which means that p is a node. If so, then $f_2(1, z)$ has two roots, say α_1, α_2 , so it is a scalar multiple of $(z - \alpha_1)(z - \alpha_2)$. Then the partial derivative $\frac{\partial g}{\partial x}$ isn't zero at the points $p_1 = (0, \alpha_1)$ and $p'_1 = (0, \alpha_2)$. They are smooth points of C_1 . Conversely, if $C_1 \cap Z$ consists of two smooth points of C_1 , those points have multiplicity 1.

Suppose that $C_1 \cap Z$ consists of a single point of multiplicity two. We may suppose $f_2 = y^2$, and then $g(x, z) = z^2 + xh(x, z)$. The curve C_1 will be smooth at p_1 if and only if the linear term $xh(0, 0)$ of g is nonzero. The constant coefficient $h(0, 0)$ of h is the coefficient of x^3 in $f_3(x, y)$. So C_1 is smooth over p if and only if the coefficient of x^3 is nonzero – if and only if p is a cusp. \square

A singularity more complicated than a node or cusp can be resolved (made smooth) by repeating the blowing up process a finite number of times.

transcdeg

1.11 Transcendence degree

Let $k \subset K$ be a field extension. A set $\alpha = \{\alpha_1, \dots, \alpha_n\}$ of elements of K is *algebraically dependent* over k if there is a nonzero polynomial $f(x_1, \dots, x_n)$ with coefficients in k , such that $f(\alpha) = 0$. If $f(\alpha) \neq 0$ for every nonzero polynomial f with coefficients in k , the set α is *algebraically independent* over k . An infinite set is algebraically independent if every finite subset is algebraically independent.

The set consisting of a single element α_1 of K is algebraically dependent if and only if α_1 is algebraic over k , and α is *transcendental* over k if it is not algebraic over k .

A set $\alpha = \{\alpha_1, \dots, \alpha_n\}$ is a *transcendence basis* for K over k if it is a maximal algebraically independent set – if it isn't contained in a larger algebraically independent set. For example, the set of variables $\{x_1, \dots, x_n\}$ is a transcendence basis over k for the field of rational functions $k(x_1, \dots, x_n)$. The *transcendence degree* of a field extension K/k is the number of elements in a transcendence basis. Lemma 1.11.2 below shows that this number depends only on the field extension.

We use the customary notation $k(\alpha_1, \dots, \alpha_n)$ or $k(\alpha)$ for the field of fractions of the algebra $k[\alpha_1, \dots, \alpha_n]$ generated by α .

1.11.1. Lemma. *Let K/k be a field extension, let $\alpha = \{\alpha_1, \dots, \alpha_n\}$ be a subset of K that is algebraically independent over k , and let β be another element of K .*

(i) *Every element of the field $k(\alpha)$ that isn't in k is transcendental over k .*

(ii) *The set $\alpha \cup \{\beta\}$ is algebraically dependent if and only if β is algebraic over the field $k(\alpha)$.*

(iii) *The set α is a transcendence basis if and only if every element of K is algebraic over $k(\alpha)$.* \square

trdeg

1.11.2. Lemma. *Let K/k be a field extension. If K has a finite transcendence basis, then all algebraically independent subsets of K are finite, and all transcendence bases have the same number of elements.*

proof. We show that if K is algebraic over the subfield $k(\alpha_1, \dots, \alpha_s)$ and if $\beta = \{\beta_1, \dots, \beta_r\}$ is an algebraically independent set, then $r \leq s$. The fact that all transcendence bases have the same order will follow: If both α and β are transcendence bases, then $r \leq s$ and $s \leq r$.

The proof proceeds by reducing to the trivial case that β is a subset of α . Suppose that some element of β , say β_r , is not in the set α . Then the set $\beta' = \{\beta_1, \dots, \beta_{r-1}\}$ is not a transcendence basis, so K is not algebraic over $k(\beta')$. Since K is algebraic over $k(\alpha)$, there is at least one element of α , say α_s , that isn't algebraic over $k(\beta')$. We replace β_r by α_s . Then $\gamma = \beta' \cup \{\alpha_s\}$ will be an algebraically independent set of order r , and it will contain more elements of the set α than β does. Induction shows that $r \leq s$. \square

trdegalg

1.11.3. Lemma. *Let A be a domain that contains a field k , and whose fraction field K has finite transcendence degree over k . There is a transcendence basis for K/k whose elements are in A .*

proof. Let $\alpha = \{\alpha_1, \dots, \alpha_k\}$ be a set of elements of A whose elements are algebraically independent over k . If k is less than the transcendence degree d , there will be an element β of K that isn't algebraic over $k(\alpha)$. We write β as a fraction γ/δ of elements of A . Then γ and δ can't both be algebraic over $k(\alpha)$, so we can add one of the two to the set α , to obtain a larger algebraically independent set. \square

This lemma allows us to define the transcendence degree of a k -algebra A that is a domain, as the maximal number of algebraically independent elements of A , which is equal to the transcendence degree of the field of fractions of A .

dualcurve

1.12 The Dual Curve

Let C be a plane projective curve of degree two or more, and let U be the set of its smooth points. The curve will have a tangent line ℓ at a point p of U , and the tangent line corresponds to a point ℓ^* of the dual plane \mathbb{P}^* (see 1.2.4). We define a map $U \xrightarrow{t} \mathbb{P}^*$ by setting $t(p) = \ell^*$. We don't try to define this map at the singular points of C .

Let U^* denote the image of U in \mathbb{P}^* . The points of U^* correspond to the tangent lines at the smooth points of C . With $f_i = \frac{\partial f}{\partial x_i}$, the tangent line ℓ at a smooth point $x = (x_0, x_1, x_2)$ of C , is the line $s_0x_0 + s_1x_1 + s_2x_2 = 0$, where $s_i = f_i(x)$. Therefore ℓ^* is the point

$$(1.12.1) \quad (s_0, s_1, s_2) = (f_0(x), f_1(x), f_2(x)).$$

The reason for the assumption that the degree isn't one is that if C were a line, U^* would be a point.

1.12.2. Theorem. *Let C be a plane curve of degree at least two, defined by an irreducible polynomial f . With notation as above, the (Zariski) closure C^* of U^* in \mathbb{P}^* is a curve in the dual space, the **dual curve**.*

1.12.3. Lemma. *Let $\varphi(s_0, s_1, s_2)$ be a homogeneous polynomial, and let*

$$g(x_0, x_1, x_2) = \varphi(f_0(x), f_1(x), f_2(x))$$

where $f_i = \frac{\partial f}{\partial x_i}$. Then $\varphi(s)$ vanishes at every point of U^* if and only if $g(x)$ vanishes at every point of U , and this happens if and only if $f(x)$ divides $g(x)$.

proof. Equation 1.12.1 shows that $\varphi(s)$ vanishes on U^* if and only if $g(x)$ vanishes on U . Since U is the complement of a finite set in C , g vanishes on U if and only if f divides g (Corollary 1.7.5). \square

proof of Theorem 1.12.2. If an irreducible homogeneous polynomial $\varphi(s_0, s_1, s_2)$ vanishes on U^* , it will be unique up to scalar factor (Corollary 1.7.5 again). Its zero locus will be the dual curve.

We show first that there is a nonzero polynomial φ , not necessarily irreducible or homogeneous, that vanishes on U^* . The field $\mathbb{C}(x_0, x_1, x_2)$ has transcendence degree 3 over \mathbb{C} . Therefore the four polynomials f_0, f_1, f_2, f are algebraically dependent. There is a nonzero polynomial $\psi(s_0, s_1, s_2, t)$ such that $\psi(f_0, f_1, f_2, f) = 0$. Since we can cancel factors of t , we may assume that ψ isn't divisible by t . Let $\varphi(s_0, s_1, s_2) = \psi(s_0, s_1, s_2, 0)$. Because t doesn't divide ψ , φ isn't zero. If x is a point of U , then $f(x) = 0$, and $\varphi(f_0, f_1, f_2) = \psi(f_0, f_1, f_2, f) = 0$. Therefore $\varphi(u)$ vanishes on U^* , as desired.

Now if $h(s)$ is the homogeneous part of φ of degree r , $h(f_0, f_1, f_2)$ will be the homogeneous part of $g(x) = \varphi(f_0, f_1, f_2)$ of degree $r(d-1)$, and it will vanish on C (Lemma 1.3.2). So we may replace φ by one of its homogeneous parts. Finally, if φ factors in $\mathbb{C}[u]$, then $g(x)$ factors accordingly, and because f is irreducible, it will divide one of the factors of g . As Lemma 1.12.3 shows, the corresponding factor of φ will vanish on U^* . So we may replace the polynomial φ , now homogeneous, by one of its irreducible factors. \square

1.12.4. Example. Determining the defining polynomial of the dual curve C^* explicitly can be painful. The degree of C^* is often different from the degree of C , and several points of C^* may correspond to a singular point of C , and vice versa. However, the computation is simple for a conic.

Let C be the conic $f = 0$, with $f = xy + yx + xz$ (see Example 1.2.11). Let $(r, s, t) = (f_x, f_y, f_z) = (y + z, x + z, x + y)$. Then

$$(1.12.5) \quad r^2 + s^2 + t^2 - 2(x^2 + y^2 + z^2) = 2f \quad \text{and} \quad rs + st + rt - (x^2 + y^2 + z^2) = 3f$$

Setting $f = 0$ gives us the equation of the dual curve C^* :

$$(1.12.6) \quad (r^2 + s^2 + t^2) - 2(rs + st + rt) = 0.$$

It is another conic. \square

ellstarequation

dualcurvethm

phizerofdividesg

exampledulone

exampledultwo

equationdulthree

equationofc-
star

(1.12.7) a local equation for the dual curve

We label the coordinates x, y, z here, and we work in a neighborhood of a smooth point p of the curve C defined by the polynomial $f(x, y, z)$. We choose coordinates so that $p = (0, 0, 1)$, and that the tangent line ℓ at p is the line $\{y = 0\}$.

Let $\tilde{f}(x, y) = f(x, y, 1)$. In the affine x, y -plane, the point p becomes $\tilde{p} = (0, 0)$, and by our choice of tangent line, $\tilde{f}(0, 0) = 0$ and $\frac{\partial \tilde{f}}{\partial y}(0, 0) \neq 0$. This allows us to solve the equation $\tilde{f} = 0$ for y as an analytic function $y(x)$, defined for small x . Let $y'(x)$ denote the derivative $\frac{dy}{dx}$. Then $y(0) = 0$ because \tilde{p} is a point of C , and $y'(0) = 0$ because the tangent line at \tilde{p} is the line $\{y = 0\}$.

Let $\tilde{p}_1 = (x_1, y_1)$ be a point of C_0 near to \tilde{p} , and let $y_1 = y(x_1)$ and $y'_1 = y'(x_1)$. The tangent line ℓ_1 at \tilde{p}_1 has the equation

localtangent

$$(1.12.8) \quad y - y_1 = y'_1(x - x_1)$$

In projective coordinates, the point ℓ_1^* of the dual plane that corresponds to ℓ_1 is

projlocaltan-
gent

$$(1.12.9) \quad (u, v, w) = (-y'_1, 1, y'_1 x_1 - y_1),$$

As x_1 varies, the solutions to this equation trace out C^* near the point ℓ^* . □

bidualC

1.12.10. Theorem. *Let C be a plane curve of degree at least 2. The **bidual** C^{**} , the dual of C^* , is the curve C .*

vopen

1.12.11. Lemma. *The set V of points p such that C is smooth at p and also C^* is smooth at $t(p)$ is the complement of a finite subset of C .* □

proof of Theorem 1.12.10. If ℓ is the tangent line to C at a point p of V , then $t(p) = \ell^*$ is a point of the image V^* of V in C^* . To avoid confusion, we denote the point ℓ^* of C^* by s , temporarily. The map $V^* \xrightarrow{t^*} \mathbb{P}^2$ analogous to t is defined by $t^*(s) = L^*$, where L is the tangent line to C^* at s . To prove the theorem, we show that $t^* \circ t$ is the identity map on V . Composition in the other order can be deduced by duality. So we must show that $L^* = p$, or that $L = p^*$. What has to be shown is:

If ℓ is the tangent line to C at a point p of V , then p^ is the tangent line to C^* at the point ℓ^* of V^* .*

We choose a second point p_1 of V , and we let p_1 approach p . Because $\ell_1^* = (f_0(p_1), f_1(p_1), f_2(p_1))$ and because the partial derivatives f_i are continuous, $\lim_{p_1 \rightarrow p} \ell_1^* = \ell^*$.

qapproachesp

1.12.12. Lemma. *With notation as above, let q be the intersection point $\ell_1 \cap \ell$. Then $\lim_{p_1 \rightarrow p} q = p$.*

figure

Let's assume the lemma for a moment. Going over to the dual space, q^* is the line through the points ℓ_1^* and ℓ^* . Since $\lim_{p_1 \rightarrow p} \ell_1^* = \ell^*$, q^* approaches the tangent line to C^* at ℓ^* . On the other hand, the lemma tells us that q^* approaches p^* . Therefore the tangent line at ℓ^* is p^* . □

proof of Lemma 1.12.12. We work analytically in a neighborhood of p . We choose coordinates as above, so that $p = (0, 0, 1)$ and that ℓ is the line $\{y = 0\}$. Let $(x_q, y_q, 1)$ be the coordinates of $q = \ell \cap \ell_1$. Since q is a point of ℓ , $y_q = 0$. Then x_q can be obtained by substituting $x = x_q$ and $y = 0$ into (1.12.8):

$$x_q = x_1 - y_1/y'_1.$$

Now: When a function has an n th order zero at a point, i.e., when it has the form $y = x^n h(x)$, where $n > 0$ and $h(0) \neq 0$, the order of zero of its derivative at that point will be $n - 1$. This is verified by differentiating $x^n h(x)$. Therefore y_1/y'_1 tends to zero with x_1 , $\lim_{p_1 \rightarrow p} x_q = 0$, and $\lim_{p_1 \rightarrow p} q = (0, 0, 1) = p$. □

A *bitangent* to a curve C is a line that is tangent to C at distinct smooth points p and q , and a bitangent is *ordinary* if neither p nor q is a flex point, and if ℓ isn't tangent to C at a third point. As before, a flex point p is ordinary if the curve and its tangent line have a contact of order three at p .

dualcusp

1.12.13. Proposition. *Let C be a smooth curve. If a line ℓ is an ordinary bitangent, then ℓ^* is a node of the dual curve C^* . If a line ℓ is tangent to C at an ordinary flex point p , then ℓ^* is a cusp of C^* . The dual curve is smooth at points that aren't the images of flexes or bitangents.*

proof. Let p be a smooth point of C . We set $z = 1$ and choose affine coordinates so that p is the origin and the tangent line ℓ at p is $\{y = 0\}$. Let $\tilde{f}(x, y) = f(x, y, 1)$. We solve $\tilde{f} = 0$ for $y = y(x)$ as analytic function of x near zero, as before. The tangent line ℓ_1 to C at a nearby point $p_1 = (x_1, y_1)$ has the equation (1.12.9).

If ℓ is an ordinary tangent line, y_1 will have a zero of order 2 at the origin. In that case $u = -y'_1$ has a simple zero. The implicit function theorem tells us that we may solve for x_1 as a function of u . Then $w = y'_1 x_1 - y_1$ becomes a function of u , and C^* is smooth at the origin.

If ℓ is an ordinary bitangent, tangent to C at the points p and q , the reasoning given for an ordinary tangent shows that the images in C^* of small neighborhoods of p and q in C will be smooth at ℓ^* , and their tangent directions p^* and q^* will be distinct. Therefore the blowup of C^* at ℓ^* consists of two smooth points, so ℓ^* is a node (see Proposition 1.10.9).

Suppose p is an ordinary flex. As before, we solve for y_1 as a function of x_1 near zero. Then y_1 has a triple zero at $x_1 = 0$. The coordinates u and w in (1.12.8) have zeros of orders 2 and 3, respectively. We blow up C^* , substituting $u = u$ and $w = ut$. Then t has a zero of order 1, so s is a function of t near zero. The blowup C^*_1 is smooth at the point $u = t = 0$, and Proposition 1.10.9 shows that ℓ^* is a cusp. \square

The next corollary wouldn't be easy to prove directly.

finitebitan-
gents

1.12.14. Corollary. *A plane curve has finitely many bitangents.*

This follows from the fact that the dual curve C^* has finitely many nodes. \square

classofcurve

1.12.15. Proposition. *If C is a smooth projective plane curve of degree d , the degree of its dual curve C^* is $d^* = d(d - 1)$.*

proof. When we project generically to \mathbb{P}^1 from a point q , the discriminant of the defining polynomial f with respect to z has degree $d(d - 1)$ (see Section 1.6). There will be $d(d - 1)$ ordinary tangent lines through q . This means that C^* will meet the line q^* in $d(d - 1)$ points. (This is Bézout's Theorem when one curve is a line (1.7.6). \square

dualsing

1.12.16. Corollary. *The dual curve C^* of a smooth curve C of degree $d \geq 3$ is singular.*

proof. When C is smooth, the map to C^* is defined everywhere. If C^* were smooth the inverse map $C^* \rightarrow C^{**} = C$, would also be defined everywhere. Then C and C^* would be homeomorphic. But if C^* were smooth, its Euler characteristic would be $e^* = 3d^* - d^{*2} = 2d^2 + 2d^3 - d^4$, while the Euler characteristic of C is $e = 3d - d^2$. But $e^* < e$ when $d \geq 3$. \square

plucker

(1.12.17) the Plücker formulas

We call a plane projective curve C *ordinary* if it is smooth, and if its flexes and bitangents are ordinary. It can be shown by counting constants that a generic curve is ordinary, but let's omit the proof. The *Plücker formulas* determine the number of bitangents of an ordinary curve C , something that isn't especially easy to do directly. For the proof, we make use of Bézout's Theorem.

plform

1.12.18. Theorem. Plücker Formulas. *Let C be an ordinary curve of degree $d \geq 2$, and let C^* be its dual curve. Let f and b denote the numbers of flex points and bitangents of C , and let δ^* and κ^* denote the numbers of nodes and cusps of C^* , respectively.*

(i) *The dual curve C^* has no flexes or bitangents. Its singularities are nodes and cusps.*

(ii) *The degree of C^* is $d^* = d(d - 1)$.*

(iii) *$f = \kappa^* = 3d(d - 2)$, and $b = \delta^* = \frac{1}{2}d(d - 2)(d^2 - 9)$*

proof. 1.12.18 (i) A bitangent or a flex on C^* would produce a singularity on the smooth curve $C = C^{**}$.

(ii) This is Proposition 1.12.15.

(iii) The facts that $f = \kappa^*$ and $b = \delta^*$ are dealt with in Proposition 1.12.13. Bézout's Theorem counts the flex points (see (1.7.7)).

To determine the number of bitangents, we use the Euler characteristic $e(C^*)$ of the singular dual curve C^* . This Euler characteristic is computed by the formula (1.8.2) in terms of a triangulation, as for smooth curves. We compute this Euler characteristic in two ways: First, the map $C \rightarrow C^*$ is defined everywhere. If ℓ is a bitangent line, tangent to C at p_1 and p_2 , then the images of p_1 and p_2 are equal to ℓ^* . The map is bijective at all points except at the images of the bitangent lines, though when p is a flex point with tangent line ℓ , C^* will be singular at the image ℓ^* . Therefore $e(C^*) = e(C) - \delta^*$. Since $e(C) = 3d - d^2$ (1.8.5),

$$(1.12.19) \quad e(C^*) = 3d - d^2 - \delta^*$$

Next, we compute the Euler characteristic $e(C^*)$ by analyzing a generic projection $C^* \xrightarrow{\pi} \mathbb{P}^1$, as described in the previous section. Such a projection will represent C^* as a covering of \mathbb{P}^1 of degree d^* .

We write the defining equation of C^* as a polynomial in x , say

$$g = b_0x^{d^*} + b_1x^{d^*-1} + \cdots + b_{d^*}$$

The discriminant $\text{Discr}_x(g)$ will be a polynomial of degree $d^*(d^* - 1)$ in y, z that vanishes at the images of smooth branch points, the nodes and the cusps, of C^* . Lemma 1.6.6 determines the orders of vanishing. Since the discriminant has degree $d^*(d^* - 1)$, the lemma gives us the formula $d^*(d^* - 1) = \beta^* + 2\delta^* + 3\kappa^*$, or

$$\beta^* = d^*(d^* - 1) - 2\delta^* - 3\kappa^*$$

where β^* denotes the number of smooth branch points.

The fibre over a point of \mathbb{P}^1 at which the discriminant is zero will have order $d^* - 1$. Each zero decreases the Euler characteristic by 1.

Therefore the Euler characteristic of C^* is

$$(1.12.20) \quad e(C^*) = d^*e(\mathbb{P}^1) - (\beta^* + \delta^* + \kappa^*) = 2d^* - d^*(d^* - 1) + \delta^* + 2\kappa^*$$

Substituting the values of d^* , δ^* , and κ^* in terms of d yields the formula for δ^* . □

1.12.21. Examples. (i) All curves of degree 2 and all smooth curves of degree 3 are ordinary.

(ii) A curve of degree 2 has no flexes and no bitangents. Its dual curve has degree 2.

(iii) A smooth curve of degree 3 has 9 flexes and no bitangents. Its dual curve has degree 6.

(iv) An ordinary curve C of degree 4 has 24 flexes and 28 bitangents. Its dual curve has degree 12. □

1.12.22. Example. (*the dual of a cuspidal cubic*) The dual of a smooth cubic curve has degree 6, and it has 9 cusps. It would be painful to compute its equation. We'll carry the computation out for a cubic with a cusp instead. Let C be the curve defined by the irreducible polynomial $f = y^2z - x^3$. Its only singularity is a cusp at $(0, 0, 1)$. The Hessian matrix is

$$H = \begin{pmatrix} -6x & 0 & 0 \\ 0 & 2z & 2y \\ 0 & 2y & 0 \end{pmatrix}$$

and the Hessian determinant $h = -24xy^2$. The common zeros of f and h are the cusp point $(0, 0, 1)$ and the point $(0, 1, 0)$. So $(0, 1, 0)$ is the unique flex point. We can guess the result. The dual curve C^* will have one cusp and one flex, the images of the flex and the cusp of C , respectively. It must be a cuspidal cubic too.

We scale the partial derivatives of f . Let $u = x^2 = f_x/3$, $v = yz = f_y/2$, and $w = y^2 = f_z$. The recipe used in the proof of Theorem 1.12.2 for finding the polynomial that defines C^* is to find a relation among f, u, v, w , and then set $f = 0$. We compute:

$$f^2 = y^4z^2 - 2x^3y^2z + x^6 = v^2w - 2uv(xy) + u^3.$$

Working modulo f , we have $v^2w + u^3 = 2uv(xy)$. Squaring both sides,

$$v^4w^2 + 2u^3v^2w + u^6 = 4u^2v^2(x^2y^2) = 4u^3v^2w,$$

or $(v^2w - u^3)^2 = 0$. The zero locus of the irreducible polynomial $v^2w - u^3$ is the dual curve. □